

NATURALLY FULL FUNCTORS IN NATURE

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ABSTRACT. We introduce and discuss the notion of naturally full functor. The definition is similar to the definition of separable functor: a naturally full functor is a functorial version of a full functor, while a separable functor is a functorial version of a faithful functor. We study general properties of naturally full functors. We also discuss when functors between module categories and between categories of comodules over a coring are naturally full.

INTRODUCTION

Separable functors were introduced by Năstăsescu, Van den Bergh and Van Oystaeyen [12]. The terminology is inspired by the result that the restriction of scalars functor associated to a morphism of rings is a separable functor if and only if the corresponding ring extension is separable. Separable functors have been studied extensively during the passed decade, we refer to [8] for a detailed discussion of results and applications. The original definition from [12] can be restated in a more categorical way as follows: to a functor $F : \mathcal{C} \rightarrow \mathcal{D}$, we can associate a natural transformation $\mathcal{F} : \text{Hom}_{\mathcal{C}}(\bullet, \bullet) \rightarrow \text{Hom}_{\mathcal{D}}(F(\bullet), F(\bullet))$, mapping f to $F(f)$; F is called separable if \mathcal{F} has a left inverse. Recall that F is called faithful if every $\mathcal{F}_{C,C'}$ is injective, i.e. it has a left inverse in the category of sets. Therefore a separable functor is faithful; in fact we could call a separable functor naturally faithful, in the sense that every $\mathcal{F}_{C,C'}$ has a left inverse that is functorial in C and C' .

The aim of this note is to study the notion of naturally full functors; a functor F is naturally full if \mathcal{F} has a right inverse; if F is full, then every $\mathcal{F}_{C,C'}$ is surjective, and has a right inverse - F is naturally full when this right inverse is functorial.

In Section 2, we study general properties of naturally full functors; some of these properties are analogous to some properties of separable functors: We have full versions of the Rafael and Rafael-Frobenius Theorems telling when a functor having an adjoint (resp. a Frobenius functor) is naturally full.

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In Section 3, we study some particular examples. From an algebraic point of view, the first example to look at is extension and restriction of scalars. The restriction of scalars functors is naturally full if and only if it is full. The same result is not true for the extension of scalars functor, see Example 3.3. We can extend our results to module categories connected by a bimodule, see Section 3.2. In Sections 3.3 and 3.4, we look at categories of comodules over a coring.

1. PRELIMINARIES

We use the following conventions. For an object V in a category, the identity morphism $V \rightarrow V$ is denoted by V . A functor is assumed to be covariant. By a ring, we will always mean a ring with unit. For a ring R , \mathcal{M}_R (resp. ${}_R\mathcal{M}$, ${}_R\mathcal{M}_R$) denotes the category of right R -modules (resp. left R -modules, R -bimodules).

1.1. Adjoint functors. Let (F, G) be a pair of adjoint functors between two categories \mathcal{C} and \mathcal{D} . This means that there exist two natural transformations $\eta : 1_{\mathcal{C}} \rightarrow GF$ and $\varepsilon : FG \rightarrow 1_{\mathcal{D}}$, called the unit and counit of the adjunction, such that

$$(1) \quad G(\varepsilon_D) \circ \eta_{G(D)} = I_{G(D)} \text{ and } \varepsilon_{F(C)} \circ F(\eta_C) = I_{F(C)}$$

for all $C \in \mathcal{C}$ and $D \in \mathcal{D}$. Then there exists a natural isomorphism

$$(2) \quad \varphi_{C,D} : \text{Hom}_{\mathcal{D}}(FC, D) \rightarrow \text{Hom}_{\mathcal{C}}(C, GD), \quad \varphi_{C,D}(f) = G(f) \circ \eta_C$$

with inverse

$$(3) \quad \psi_{C,D} = \varphi_{C,D}^{-1} : \text{Hom}_{\mathcal{C}}(C, GD) \rightarrow \text{Hom}_{\mathcal{D}}(FC, D), \quad \psi_{C,D}(g) = \varepsilon_D \circ F(g)$$

for any $C \in \mathcal{C}$ and $D \in \mathcal{D}$. Conversely, η and ε are recovered from φ and ψ by

$$\eta_C = \varphi_{C,FC}(FC), \quad \varepsilon_D = \psi_{GD,D}(GD).$$

We also have isomorphisms (see for example [8, Proposition 11]):

$$\alpha : \underline{\text{Nat}}(GF, 1_{\mathcal{C}}) \rightarrow \underline{\text{Nat}}(\text{Hom}_{\mathcal{D}}(F(\bullet), F(\bullet)), \text{Hom}_{\mathcal{C}}(\bullet, \bullet))$$

$$\beta : \underline{\text{Nat}}(1_{\mathcal{D}}, FG) \rightarrow \underline{\text{Nat}}(\text{Hom}_{\mathcal{C}}(G(\bullet), G(\bullet)), \text{Hom}_{\mathcal{D}}(\bullet, \bullet))$$

defined as follows:

$$(4) \quad \alpha(\nu)_{C,C'}(g) = \nu_{C'} \circ G(g) \circ \eta_C$$

for all $\nu \in \underline{\text{Nat}}(GF, 1_{\mathcal{C}})$, $C, C' \in \mathcal{C}$ and $g : F(C) \rightarrow F(C')$ in \mathcal{D} with inverse

$$(5) \quad \alpha^{-1}(\mathcal{P})_C = \mathcal{P}_{GFC,C}(\varepsilon_{FC})$$

for any $\mathcal{P} \in \underline{\text{Nat}}(\text{Hom}_{\mathcal{D}}(F(\bullet), F(\bullet)), \text{Hom}_{\mathcal{C}}(\bullet, \bullet))$ and $C \in \mathcal{C}$ and respectively,

$$(6) \quad \beta(\xi)_{D,D'}(f) = \varepsilon_{D'} \circ F(f) \circ \xi_D$$

for any $\xi \in \underline{\text{Nat}}(1_{\mathcal{D}}, FG)$, $D, D' \in \mathcal{D}$ and $f : G(D) \rightarrow G(D')$ in \mathcal{C} with inverse

$$(7) \quad \beta^{-1}(\mathcal{T})_D = \mathcal{T}_{D, FGD}(\eta_{GD})$$

for any $\mathcal{T} \in \underline{\text{Nat}}(\text{Hom}_{\mathcal{C}}(G(\bullet), G(\bullet)), \text{Hom}_{\mathcal{D}}(\bullet, \bullet))$ and $D \in \mathcal{D}$. Assume now that G is a left adjoint of F and let

$$\mu : GF \rightarrow 1_{\mathcal{C}} \text{ and } \chi : 1_{\mathcal{D}} \rightarrow FG$$

be the counit and unit of the adjunction (G, F) . By [8, Proposition 10], there exist one-to-one correspondences between the following classes

$$\underline{\text{Nat}}(GF, 1_{\mathcal{C}}) \cong \underline{\text{Nat}}(G, G) \cong \underline{\text{Nat}}(F, F) \cong \underline{\text{Nat}}(1_{\mathcal{D}}, FG)$$

Let us describe these correspondences: for any natural transformation $\nu : GF \rightarrow 1_{\mathcal{C}}$ there exist unique natural transformations $\alpha : G \rightarrow G$, $\beta : F \rightarrow F$, such that

$$(8) \quad \mu_C \circ \alpha_{FC} = \nu_C = \mu_C \circ G(\beta_C);$$

for any natural transformation $\xi : 1_{\mathcal{D}} \rightarrow FG$, there exist unique natural transformations $\alpha : G \rightarrow G$, $\beta : F \rightarrow F$, such that

$$(9) \quad \beta_{GD} \circ \chi_D = \xi_D = F(\alpha_D) \circ \xi_D.$$

Recall that a functor F is called Frobenius if it has a right adjoint G which is also a left adjoint. In this case, there exist four functors ε , η , ξ and μ satisfying the properties given above.

1.2. Separable functors. Separable functors were introduced in [12]; several applications have appeared in the literature, see [8]. The definition of separable functor can be formulated in the following way: for a functor $F : \mathcal{C} \rightarrow \mathcal{D}$, we gave two functors

$$\text{Hom}_{\mathcal{C}}(\bullet, \bullet), \text{Hom}_{\mathcal{D}}(F(\bullet), F(\bullet)) : \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \underline{\text{Sets}}$$

and a natural transformation

$$\mathcal{F} : \text{Hom}_{\mathcal{C}}(\bullet, \bullet) \rightarrow \text{Hom}_{\mathcal{D}}(F(\bullet), F(\bullet)), \quad \mathcal{F}_{C, C'}(f) = F(f)$$

for any $f : C \rightarrow C'$ in \mathcal{C} . The natural transformation \mathcal{F} will play a key role in the sequel. F is called *separable* if \mathcal{F} splits, that is, we have a natural transformation

$$\mathcal{T} : \text{Hom}_{\mathcal{D}}(F(\bullet), F(\bullet)) \rightarrow \text{Hom}_{\mathcal{C}}(\bullet, \bullet)$$

such that

$$\mathcal{T} \circ \mathcal{F} = \mathbf{1}_{\text{Hom}_{\mathcal{C}}(\bullet, \bullet)}$$

If F is separable, then for all $C, C' \in \mathcal{C}$, the map $\mathcal{F}_{C, C'} : \text{Hom}_{\mathcal{C}}(C, C') \rightarrow \text{Hom}_{\mathcal{D}}(F(C), F(C'))$ is injective, since it has a left inverse, and it follows that F is a faithful functor. Since an injective map between sets has a left inverse, we can restate the definitions of separable and faithful functors in the following way: F is faithful if every $\mathcal{F}_{C, C'}$ has a left inverse; F is separable if this left inverse can be chosen to be natural in C and C' . Perhaps it

would be better, at least from the categorical point of view, to call separable functors naturally faithful functors. Observe that a faithful functor is not necessarily separable: if K/L is a purely inseparable field extension, then the restriction of scalars functor is faithful, but not separable.

Now recall the dual notion of a faithful functor: F is called a full functor if every $\mathcal{F}_{C,C'}$ is surjective, or, equivalently, has a right inverse. As far as we know, there are not many characterizations of full functors available in the literature. In the case where F is the restriction of scalars functor, we have the following result (see for example [14, Proposition XI.1.1 and XI.1.2]). For $M \in {}_S\mathcal{M}_S$, let $M^S = \{m \in M \mid sm = ms, \text{ for all } s \in S\}$ be the set of S -invariant elements of M .

Theorem 1.1. *Let $\varphi : R \rightarrow S$ be a ring morphism, and $\varphi_* : {}_S\mathcal{M} \rightarrow {}_R\mathcal{M}$ the restriction of scalars functor. The following statements are equivalent:*

- (1) *The restriction of scalars functor $\varphi_* : {}_S\mathcal{M} \rightarrow {}_R\mathcal{M}$ is full;*
- (2) *$\varphi : R \rightarrow S$ is an epimorphism of rings i.e. it is an epimorphism in the category of unital rings;*
- (3) *$M^R = M^S$, for any $M \in {}_S\mathcal{M}_S$;*
- (4) *$1_S \otimes_R 1_S$ is a separability idempotent for the extension S/R , i.e. $s \otimes_R 1_S = 1_S \otimes_R s$, for any $s \in S$;*
- (5) *the map*

$$\varepsilon_S : S \otimes_R S \rightarrow S, \quad \varepsilon_S(s_1 \otimes_R s_2) = s_1 s_2$$

is injective (hence an isomorphism in ${}_S\mathcal{M}_S$);

- (6) *the counit of the adjunction $(S \otimes_R \bullet, \varphi_*)$*

$$\varepsilon_N : S \otimes_R N \rightarrow N, \quad \varepsilon_N(s \otimes_R n) = sn$$

is an isomorphism of left S -modules, for all $N \in {}_S\mathcal{M}$.

The fact that the restriction of scalars functor $\varphi_* : {}_S\mathcal{M} \rightarrow {}_R\mathcal{M}$ is a full functor explicitly means that

$${}_R\text{Hom}(M, N) = {}_S\text{Hom}(M, N)$$

for any $M, N \in {}_S\mathcal{M}$, i.e. any left R -linear map between two left S -modules is also left S -linear.

2. NATURALLY FULL FUNCTORS

We keep the notation of Section 1. Following the philosophy of Section 1, we introduce the notion of naturally full functor.

Definition 2.1. A functor F is called a naturally full functor, if \mathcal{F} cosplits. This means that there exists a natural transformation

$$\mathcal{P} : \text{Hom}_{\mathcal{D}}(F(\bullet), F(\bullet)) \rightarrow \text{Hom}_{\mathcal{C}}(\bullet, \bullet)$$

such that

$$(10) \quad \mathcal{F} \circ \mathcal{P} = \mathbf{1}_{\text{Hom}_{\mathcal{D}}(F(\bullet), F(\bullet))}$$

Remarks 2.2. 1. The fact that \mathcal{P} is a natural transformation means the following: for any $X, Y, Z, T \in \mathcal{C}$ and $f : X \rightarrow Y$, $h : Z \rightarrow T$ in \mathcal{C} and $g : F(Y) \rightarrow F(Z)$ in \mathcal{D} , we have

$$(11) \quad \mathcal{P}_{X,T}(F(h) \circ g \circ F(f)) = h \circ \mathcal{P}_{Y,Z}(g) \circ f.$$

(10) can be rewritten as

$$(12) \quad F(\mathcal{P}_{C,C'}(u)) = u$$

for all $C, C' \in \mathcal{C}$ and $u : FC \rightarrow FC'$ in \mathcal{D} .

2. A naturally full functor is always full, but the converse is not true in general. For a counterexample, see Example 3.3.

3. The following conditions are equivalent:

- (1) F is fully faithful;
- (2) F is separable and naturally full.

4. Combining 3. with the well-known result that a functor is an equivalence if and only if it is fully faithful and surjective on objects, we find that the following conditions are equivalent:

- (1) F is a category equivalence;
- (2) F is fully faithful and surjective on objects;
- (3) F is separable, naturally full and surjective on objects.

We will now present some general properties of naturally functors. The proofs are sometimes similar to corresponding proofs for properties of separable functors, see [8].

Proposition 2.3. *Consider functors $G : \mathcal{C} \rightarrow \mathcal{D}$ and $H : \mathcal{D} \rightarrow \mathcal{E}$.*

- (1) *If G and H are naturally full, then $H \circ G$ is also naturally full.*
- (2) *If $H \circ G$ is a naturally full and H is faithful, then G is naturally full.*

Proof. 1) Obvious: if \mathcal{P}^G and \mathcal{P}^H are right inverses of \mathcal{G} and \mathcal{H} , then $\mathcal{P}^G \circ \mathcal{P}^H$ is a right inverse of $\mathcal{H} \circ \mathcal{G}$.

2) Let $\mathcal{P}^{H \circ G}$ be a right inverse of $\mathcal{H} \circ \mathcal{G}$. Then

$$\mathcal{H} \circ \mathcal{G} \circ \mathcal{P}^{H \circ G} \circ \mathcal{H} = \mathcal{H},$$

and from the fact that H is faithful, it follows that $\mathcal{P}^{H \circ G} \circ \mathcal{H}$ is a right inverse of \mathcal{G} , as needed. \square

Proposition 2.4. *Consider functors $F : \mathcal{D} \rightarrow \mathcal{C}$, $G : \mathcal{C} \rightarrow \mathcal{D}$ and $H : \mathcal{D} \rightarrow \mathcal{E}$ and assume that G is a right or left adjoint of F . If $H \circ G$ is naturally full and F is separable, then H is naturally full.*

Proof. Let $\mathcal{P}^{H \circ G}$ be a right inverse of $\mathcal{H} \circ \mathcal{G}$. If G is a right adjoint of F , then by Rafael's Theorem [13] the unit $\eta : 1_{\mathcal{C}} \rightarrow GF$ of the adjunction splits, i.e. there exists a natural transformation $\nu : GF \rightarrow 1_{\mathcal{C}}$ such that $\nu_C \circ \eta_C = C$, for all $C \in \mathcal{C}$. We now define a natural transformation \mathcal{P}^H using (4):

$$\begin{aligned} \mathcal{P}_{C,C'}^H(f) &= \alpha(\nu)_{C,C'} \{ \mathcal{P}^{H \circ G}[H(\eta_{C'}) \circ f \circ H(\nu_C)] \} \\ &= \nu_{C'} \circ G \{ \mathcal{P}^{H \circ G}[H(\eta_{C'}) \circ f \circ H(\nu_C)] \} \circ \eta_C. \end{aligned}$$

for all $f \in \text{Hom}_{\mathcal{C}}(H(C), H(C'))$. We easily compute that

$$\begin{aligned} \mathcal{H}_{C,C'} \circ \mathcal{P}_{C,C'}^H(f) &= H(\nu_{C'}) \circ HG\{\mathcal{P}^{H \circ G}[H(\eta_{C'}) \circ f \circ H(\nu_C)]\} \circ H(\eta_C) \\ &= H(\nu_{C'}) \circ H(\eta_{C'}) \circ f \circ H(\nu_C) \circ H(\eta_C) = f, \end{aligned}$$

as needed.

The proof is similar in the situation where G is a left adjoint of F : by Rafael's Theorem, the counit $\varepsilon : GF \rightarrow 1_{\mathcal{D}}$ has a right adjoint $\xi : \varepsilon_D \circ \xi_D = D$, for all $D \in \mathcal{D}$. We then define a natural transformation \mathcal{P}^H using (6). \square

Rafael's Theorem [13] provides an easy characterization of the separability of a functor having an adjoint. Similar characterizations can be formulated for full and faithful functor (Proposition 2.5) and for naturally full functors (Theorem 2.6).

Proposition 2.5. *Let $G : \mathcal{D} \rightarrow \mathcal{C}$ be a right adjoint of $F : \mathcal{C} \rightarrow \mathcal{D}$.*

- (1) *F is faithful if and only if η_C is a monomorphism in \mathcal{C} , for all $C \in \mathcal{C}$;*
- (2) *F is full if and only if η_C cosplits in \mathcal{C} , for all $C \in \mathcal{C}$, that is, there exists $\nu_C \in \mathcal{C}$ such that $\nu_C \circ \eta_C = GFC$;*
- (3) *G is faithful if and only if ε_D is a epimorphism in \mathcal{D} , for all $D \in \mathcal{D}$;*
- (4) *G is full if and only if ε_D splits in \mathcal{D} , for all $D \in \mathcal{D}$, that is, there exists $\xi_D \in \mathcal{D}$ such that $\xi_D \circ \varepsilon_D = FGD$.*

Proof. For any $C, C' \in \mathcal{C}$, we consider the composition

$$\Omega_{C,C'} = \varphi_{C,FC'} \circ \mathcal{F}_{C,C'} : \text{Hom}_{\mathcal{C}}(C, C') \rightarrow \text{Hom}_{\mathcal{C}}(C, GFC'),$$

where φ is the natural isomorphism defined in Section 1.1. We easily compute that

$$\Omega_{C,C'}(f) = \varphi_{C,C'}(F(f)) = GF(f) \circ \eta_C = \eta_{C'} \circ f.$$

Since φ is an isomorphism, injectivity or surjectivity of $\mathcal{F}_{C,C'}$ is equivalent to injectivity or surjectivity of $\Omega_{C,C'}$.

Now $\Omega_{C,C'}$ is injective if and only if $\eta_{C'}$ is a monomorphism, and this proves the first statement.

If $\Omega_{GFC,C} : \text{Hom}_{\mathcal{C}}(GFC, C) \rightarrow \text{Hom}_{\mathcal{C}}(GFC, GFC)$ is surjective, then there exists a $\nu_C : GFC \rightarrow C$ such that

$$\Omega_{GFC,C}(\nu_C) = \eta_C \circ \nu_C = GFC$$

Conversely, assume that for every $C \in \mathcal{C}$, there exists ν_C such that $\eta_C \circ \nu_C = GFC$. For every $f : C \rightarrow GFC'$ in \mathcal{C} , we have

$$\Omega_{C,C'}(\nu_{C'} \circ f) = \eta_{C'} \circ \nu_{C'} \circ f = f$$

and it follows that $\Omega_{C,C'}$ is surjective. This finishes the proof of the second statement.

The two remaining properties are the duals of the first and the second one. \square

Our next result is Rafael's Theorem for naturally full functors.

Theorem 2.6. *Let (F, G) be an adjoint pair of functors between \mathcal{C} and \mathcal{D} , with unit η and counit ε .*

1. F is naturally full if and only if $\eta : 1_{\mathcal{C}} \rightarrow GF$ cosplits, i.e. there exists a natural transformation $\nu : GF \rightarrow 1_{\mathcal{C}}$ such that

$$(13) \quad \eta_C \circ \nu_C = GFC$$

for all $C \in \mathcal{C}$. In this case, ε_{FC} is an isomorphism in \mathcal{D} with inverse $F(\eta_C)$.

2. G is naturally full if and only if $\varepsilon : FG \rightarrow 1_{\mathcal{D}}$ splits, i.e. there exists a natural transformation $\xi : 1_{\mathcal{D}} \rightarrow FG$ such that

$$(14) \quad \xi_D \circ \varepsilon_D = FGD$$

for all $D \in \mathcal{D}$. In this case, $G(\varepsilon_D)$ is an isomorphism in \mathcal{C} with inverse η_{GD} .

Proof. 1. Assume first that F is naturally full and let \mathcal{P} be a right inverse of \mathcal{F} . Let $\nu = \alpha^{-1}(\mathcal{P})$ be the natural transformation given by (5):

$$\nu_C : GFC \rightarrow C, \quad \nu_C = \mathcal{P}_{GFC, C}(\varepsilon_{FC})$$

We consider the natural isomorphism ψ given by (3). Then we have

$$\begin{aligned} \psi(\eta_C \circ \nu_C) &\stackrel{(3)}{=} \varepsilon_{FC} \circ F(\eta_C \circ \nu_C) = \varepsilon_{FC} \circ F(\eta_C) \circ F(\mathcal{P}_{GFC, C}(\varepsilon_{FC})) \\ &\stackrel{(12)}{=} \varepsilon_{FC} \circ F(\eta_C) \circ \varepsilon_{FC} \stackrel{(1)}{=} \varepsilon_{FC} = \varepsilon_{FC} \circ F(Id_{GFC}) \stackrel{(3)}{=} \psi(Id_{GFC}) \end{aligned}$$

From the fact that ψ is a natural isomorphism, we obtain that $\eta_C \circ \nu_C = Id_{GFC}$, for all $C \in \mathcal{C}$. Furthermore,

$$GFC = \eta_C \circ \nu_C = \eta_C \circ \mathcal{P}_{GFC, C}(\varepsilon_{FC}) \stackrel{(11)}{=} \mathcal{P}_{GFC, GFC}(F(\eta_C) \circ \varepsilon_{FC}).$$

Applying F and using (12), we obtain that $F(\eta_C) \circ \varepsilon_{FC} = Id_{FGFC}$. Combining this with (1), we see that ε_{FC} and $F(\eta_C)$ are each others inverses. Conversely, let $\nu : GF \rightarrow 1_{\mathcal{C}}$ be such that $\eta_C \circ \nu_C = GFC$ for all $C \in \mathcal{C}$ and let $\mathcal{P} = \alpha(\nu)$ be the natural transformation given by (4), i.e.

$$\mathcal{P}_{C, C'}(f) = \nu_{C'} \circ G(f) \circ \eta_C$$

for all $C, C' \in \mathcal{C}$ and $f : F(C) \rightarrow F(C')$ in \mathcal{D} . We have to show that $F(\mathcal{P}_{C, C'}(f)) = f$. We easily compute that

$$\begin{aligned} F(\nu_{C'} \circ G(f) \circ \eta_C) &= Id_{FC'} \circ F(\nu_{C'} \circ G(f) \circ \eta_C) \\ &\stackrel{(1)}{=} \varepsilon_{FC'} \circ F(\eta_{C'}) \circ F(\nu_{C'} \circ G(f) \circ \eta_C) \\ &= \varepsilon_{FC'} \circ F(\eta_{C'} \circ \nu_{C'} \circ G(f) \circ \eta_C) \\ &= \varepsilon_{FC'} \circ F(G(f) \circ \eta_C) \stackrel{(2)}{=} (\psi \circ \varphi)(f) = f, \end{aligned}$$

as needed.

2. The second statement follows from the first one by duality arguments. \square

The Rafael criterion for separability simplifies further if we consider a Frobenius functor. In Proposition 2.7, we give necessary and sufficient conditions for the natural fullness of a Frobenius functor.

Proposition 2.7. *Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a Frobenius functor, with left and right adjoint G . We use the same notation as in Section 1 for the unit and counit of the two adjunctions. Then the following statements are equivalent.*

- (1) *F is a naturally full;*
- (2) *there exists a natural transformation $\alpha : G \rightarrow G$ such that*

$$(15) \quad \eta_C \circ \mu_C \circ \alpha_{FC} = GFC,$$
for all $C \in \mathcal{C}$;
- (3) *there exists a natural transformation $\beta : F \rightarrow F$ such that*

$$(16) \quad \eta_C \circ \mu_C \circ G(\beta_C) = GFC,$$
for all $C \in \mathcal{C}$;
- (4) *there exists a natural transformation $\tilde{\alpha} : G \rightarrow G$ such that*

$$(17) \quad \tilde{\alpha}_{FC} \circ \eta_C \circ \mu_C = GFC,$$
for all $C \in \mathcal{C}$;
- (5) *there exists $\tilde{\beta} : F \rightarrow F$ a natural transformation such that*

$$(18) \quad G(\tilde{\beta}_C) \circ \eta_C \circ \mu_C = GFC,$$
for all $C \in \mathcal{C}$.

In this case, $\eta_C \circ \mu_C$ is an isomorphism in \mathcal{C} and

$$\alpha_{FC} = \tilde{\alpha}_{FC} = G(\beta_C) = G(\tilde{\beta}_C),$$

Proof. 1. Apply first Theorem 2.6 to the adjunction (F, G) and then (8) to the adjunction (G, F) , in order to describe all natural transformations ν in terms of α and β . This entails the equivalence of 1), 2) and 3).

Then we view F as a right adjoint of G and apply Theorem 2.6 and (9) to the adjunction (G, F) ; we obtain a description of all natural transformations ξ in terms of $\tilde{\alpha}$ and $\tilde{\beta}$, and we find the equivalence 1), 4) and 5). \square

3. APPLICATIONS AND EXAMPLES

3.1. Extension and restriction of scalars. Let $\varphi : R \rightarrow S$ be a morphism of rings. The restriction of scalars functor $\varphi_* : {}_S\mathcal{M} \rightarrow {}_R\mathcal{M}$ is a right adjoint of the extension of scalars functor $\varphi^* = S \otimes_R \bullet : {}_R\mathcal{M} \rightarrow {}_S\mathcal{M}$. The unit and counit of the adjunction are $\eta_M = \varphi \otimes_R M : M \rightarrow S \otimes_R M$ and $\varepsilon_N : S \otimes_R N \rightarrow N$, $\varepsilon_N(s \otimes_R n) = sn$. It is well-known and easy to prove, see for example [8], that

$$\underline{\text{Nat}}(\varphi_* \circ \varphi^*, 1_{{}_R\mathcal{M}}) \simeq {}_R\text{Hom}(S, R)_R \text{ and } \underline{\text{Nat}}(1_{{}_S\mathcal{M}}, \varphi^* \circ \varphi_*) \simeq (S \otimes_R S)^S.$$

The natural transformation ν^E corresponding to $E \in {}_R\text{Hom}(S, R)_R$ is given by

$$\nu_M^E : S \otimes_R M \rightarrow M, \quad \nu_M^E(s \otimes_R m) = E(s)m.$$

The natural transformation ξ^e corresponding to $e = \sum e^1 \otimes e^2 \in (S \otimes_R S)^S$ is given by

$$(19) \quad \xi_N^e : N \rightarrow S \otimes_R N, \quad \xi_N^e(n) = \sum e^1 \otimes_R e^2 n.$$

Proposition 3.1. *Let $\varphi : R \rightarrow S$ be a ring morphism.*

1. *The restriction of scalars functor φ_* is naturally full if and only if it is full.*
2. *The following statements are equivalent:*
 - (1) *The extension of scalars functor φ^* is naturally full;*
 - (2) *there exists an $E \in {}_R\text{Hom}(S, R)_R$ such that $\varphi \circ E = S$;*
 - (3) *there exists a central idempotent e of R such that $S \cong Re$ and $\varphi : R \rightarrow S \cong Re$ is the projection $\varphi(r) = re$.*

Proof. 1. If φ_* is full, then it follows from the equivalence (1) \iff (6) in Theorem 1.1 that the counit ε is a natural isomorphism; in particular ε has a left inverse, and it follows from Theorem 2.6 that φ_* is naturally full.

2. (1) \implies (2). By Theorem 2.6, η has a right inverse ν ; take the corresponding $E \in {}_R\text{Hom}(S, R)_R$. For all $s \in S$, we have

$$s = (\eta_R \circ \nu_M^E)(s) = \eta_R(E(s)) = \varphi(E(s))$$

(2) \implies (1). For all $M \in {}_R\mathcal{M}$, $m \in M$ and $s \in S$, we compute

$$\begin{aligned} (\eta_M \circ \nu_M^E)(s \otimes_R m) &= \eta_M(E(s)m) = 1_S \otimes_R E(s)m \\ &= 1_S E(s) \otimes_R m = \varphi(E(s)) \otimes_R m = s \otimes_R m. \end{aligned}$$

(2) \implies (3). Let $e = E(1_S)$. Then

$$\begin{aligned} r \cdot e &= rE(1_S) = E(r1_S) = E(\varphi(r)1_S) = E(\varphi(r)) \\ &= E(1_S\varphi) = E(1_Sr) = E(1_S)r = e \cdot r \end{aligned}$$

and

$$\begin{aligned} e^2 &= E(1_S) \cdot E(1_S) = E(E(1_S) \cdot 1_S) \\ &= E(\varphi(E(1_S))1_S) = E(1_S1_S) = E(1_S) = e, \end{aligned}$$

so e is a central idempotent in R . The restriction $\varphi' = \varphi|_{Re} : Re \rightarrow S$ is an isomorphism, since

$$E(\varphi(re)) = E(\varphi(r)\varphi(e)) = rE(\varphi(e)) = rE(1_S) = re$$

and $\varphi \circ E = S$.

(3) \implies (2) is trivial. \square

Remarks 3.2. 1) It follows from Proposition 3.1 that natural fullness of the restriction and extension of scalars functor is left-right symmetric: φ_* is naturally full if and only if ${}_*\varphi : \mathcal{M}_S \rightarrow \mathcal{M}_R$ is naturally full.

2) Recall that a map of left R -modules $f : M \rightarrow M'$ is called pure if the map

$$N \otimes_R f : N \otimes_R M \rightarrow N \otimes_R M'$$

is injective, for every $N \in \mathcal{M}_R$. If $\varphi : R \rightarrow S$ is pure as a morphism of left or right R -modules, then the map

$$R \rightarrow \{s \in S \mid s \otimes_R 1 = 1 \otimes_R s\}$$

is an isomorphism (see for example [4, Prop. 2.1]). If S is faithfully flat as a left (right) R -module, then φ is pure as morphism of left (right) R -modules. Assume that φ is left (or right) pure, and that φ_* is naturally full. Then it follows from condition (4) in Theorem 1.1 that φ is the identity, hence $R = S$.

3) Assume that φ^* is naturally full. It follows from Proposition 3.1 that, up to isomorphism, we may assume that $S = Re$, where e is a central idempotent of R . Then $S \otimes_R S = Re \otimes_R Re \cong Re$, and $e \otimes e$ is a separability idempotent of S/R . Now e is the unit element of S , so φ_* is also naturally full. Furthermore, the inclusion $i : S \rightarrow R$ is an R -bimodule map, and $(e \otimes_R e, i)$ is a Frobenius system for $R \rightarrow S$. Hence S/R is a Frobenius extension.

We are now able to give an example of a functor which is full but not naturally full.

Example 3.3. Let S be a semisimple artinian ring, T a ring, and $0 \neq L \in {}_S\mathcal{M}_T$. Consider the ring

$$R = \begin{pmatrix} S & L \\ 0 & T \end{pmatrix},$$

and the obvious projection $\varphi : R \rightarrow S$ and injection $\nu_R : S \rightarrow R$. φ is a surjective ring homomorphism, and ν_R is a left (but not right) R -module section of φ , so $\varphi_*(S)$ is projective as a left R -module. S is semisimple, so $\varphi_*(N)$ is a projective left R -module, for every $N \in {}_S\mathcal{M}$. In particular, the map $\eta_M : M \rightarrow \varphi_*(\varphi^*(M))$ cosplits in ${}_R\mathcal{M}$, so there exists a $\nu_M : \varphi_*(\varphi^*(M)) \rightarrow M$ in ${}_R\mathcal{M}$ such that $\eta_M \circ \nu_M = \varphi_*(\varphi^*(M))$. By Proposition 2.5, this means that φ^* is full. It follows from Proposition 3.1 that φ^* is not naturally full.

Remark 3.4. Consider ring homomorphisms $\alpha : A \rightarrow B$ and $\beta : B \rightarrow C$. If $\alpha_* \circ \beta_* = (\beta \circ \alpha)_*$ is naturally full, then $\beta \circ \alpha$ (and a fortiori β) is an epimorphism in the category of rings, and consequently β_* is naturally full. This can also be deduced from Proposition 2.3, since α_* is faithful. Observe that α_* is not naturally full in general: take the canonical inclusion $\alpha : \mathbb{Z} \rightarrow \mathbb{Z}[X]$ and the canonical projection $\beta : \mathbb{Z}[X] \rightarrow \mathbb{Z}[X]/(X)$. So the fact that HG is naturally full does not imply that H is naturally full.

Remark 3.5. The equivalence (1) \Leftrightarrow (4) in Theorem 1.1 can be obtained from Theorem 2.6 as follows. Let $e = \sum e^1 \otimes e^2 \in (S \otimes_R S)^S$. Then, in view of (19), we have:

$$(\xi_N^e \circ \varepsilon_N)(s \otimes_R r) = \xi_N^e(sn) = \sum e^1 \otimes_R e^2 sn.$$

Thus, ξ_N^e splits ε_N iff

$$\sum e^1 \otimes_R e^2 sn = s \otimes_R n$$

for every $N \in {}_S\mathcal{M}$, $n \in N$ and $s \in S$. Clearly this is equivalent to $1_S \otimes_R 1_S = e \in (S \otimes_R S)^S$.

3.2. Induction and coinduction. Let R and S be rings and $M \in {}_S\mathcal{M}_R$ a (S, R) -bimodule. To M we can associate a pair of adjoint functors

$$F = M \otimes_R \bullet : {}_R\mathcal{M} \rightarrow {}_S\mathcal{M} ; G = {}_S\text{Hom}(M, \bullet) : {}_S\mathcal{M} \rightarrow {}_R\mathcal{M}$$

For $Q \in {}_S\mathcal{M}$, ${}_S\text{Hom}(M, Q) \in {}_R\mathcal{M}$ via the formula $(m)(r \cdot f) = (mr)f$, for all $m \in M$, $r \in R$ and $f \in {}_S\text{Hom}(M, Q)$. The unit η and the counit ε of the adjunction are given by

$$\eta_P : P \rightarrow {}_S\text{Hom}(M, M \otimes_R P), \quad \eta_P(p)(m) = m \otimes_R p;$$

$$\varepsilon_Q : M \otimes_R {}_S\text{Hom}(M, Q) \rightarrow Q, \quad \varepsilon_Q(m \otimes_R f) = (m)f.$$

Observe that ${}_S\text{End}(M) \in {}_R\mathcal{M}_R$ via

$$(m)(r' \cdot f \cdot r) = ((mr')f)r;$$

also ${}^*M = {}_S\text{Hom}(M, S) \in {}_R\mathcal{M}_S$ via

$$(m)(r \cdot f \cdot s) = ((mr)f)s.$$

Proposition 3.6. *Let R, S, M, F, G be as above. Then*

$$\underline{\text{Nat}}(1_{{}_S\mathcal{M}}, FG) \simeq (M \otimes_R {}^*M)^S.$$

If M is finitely generated and projective as a left S -module, then

$$\underline{\text{Nat}}(GF, 1_{{}_R\mathcal{M}}) \simeq {}_R\text{Hom}_R({}_S\text{End}(M), R).$$

Proof. We refer to [7] or [8] for full detail. Let us give a sketch of the proof. The natural transformation $\xi : 1_{{}_S\mathcal{M}} \rightarrow FG$ corresponding to $\sum_i m_i \otimes_R f_i \in (M \otimes_R {}^*M)^S$ is the following:

$$(20) \quad \xi_Q : Q \rightarrow M \otimes_R {}_S\text{Hom}(M, Q), \quad (q)\xi_Q = \sum_i m_i \otimes_R (?)f_i q$$

The natural transformation $\nu : GF \rightarrow 1_{{}_R\mathcal{M}}$ corresponding to an R -bimodule map $E : {}_S\text{End}(M) \rightarrow R$ is given by

$$(21) \quad \left(\left(\sum_i g_i \otimes_R p_i \right) (\psi_{M,P}) \right) \nu_P = \sum_i (g_i)E \cdot p_i$$

where

$$\psi_{M,P} : {}_S\text{End}(M,) \otimes_R P \xrightarrow{\sim} {}_S\text{Hom}(M, M \otimes_R P)$$

$$(m) [(g \otimes_R p)\psi_{M,P}] = (m)g \otimes_R p$$

is the canonical isomorphism of left R -modules, as ${}_S M$ is finitely generated and projective. \square

Before we are able to discuss natural fullness of the induction and coinduction functor, we need a Lemma.

Lemma 3.7. *Let $M \in {}_S\mathcal{M}_R$, $m_1, \dots, m_n \in M$ and $f_1, \dots, f_n \in {}^*M$. Then*

$$(22) \quad m \otimes_R f = \sum_{i=1}^n m_i \otimes_R (?)f_i \cdot (m)f \text{ in } M \otimes_R {}_S\text{Hom}(M, Q)$$

for every $Q \in {}_S\mathcal{M}$, $f \in {}_S\text{Hom}(M, Q)$ and $m \in M$ if and only if

$$(23) \quad m \otimes_R M = \sum_{i=1}^n m_i \otimes_R (?)f_i m \text{ in } M \otimes_R {}_S\text{End}(M) \text{ for every } m \in M.$$

Proof. Taking $Q = M$ and $f = M$ in (22), we obtain (23). Conversely, assume that (23) holds. Set $A = {}_S\text{End}(M)$ and consider the canonical isomorphism:

$$\begin{aligned} \text{can} : M \otimes_R A \otimes_A {}_S\text{Hom}(M, Q) &\cong M \otimes_R {}_S\text{Hom}(M, Q) \\ \text{can}(m \otimes_R a \otimes_A f) &= m \otimes_R a \cdot f. \end{aligned}$$

By (23) we have:

$$m \otimes_R \otimes_A f = \sum_{i=1}^n m_i \otimes_R (?)f_i m \otimes_A f.$$

Applying can, we find (22). \square

Theorem 3.8. *Let M be an (S, R) -bimodule. 1. The coinduction functor $G = {}_S\text{Hom}(M, \bullet)$ is naturally full if and only if there exists $\sum_{i=1}^n m_i \otimes_R f_i \in (M \otimes_R {}^*M)^S$ satisfying (23).*

2. Assume that $M \in {}_S\mathcal{M}$ is finitely generated and projective. Then the following assertions are equivalent.

- (1) *The induction functor $F = M \otimes_R \bullet$ is naturally full;*
- (2) *there exists $E \in {}_R\text{Hom}_R({}_S\text{End}(M), R)$ such that $\chi \circ E = {}_S\text{End}(M)$;*
- (3) *there exists a central idempotent e of R such that ${}_S\text{End}(M) \cong Re$ and $\xi : R \rightarrow {}_S\text{End}(M) \cong Re$ is the projection $\xi(r) = re$.*

Proof. 1. Let $\xi : {}_1{}_S\mathcal{M} \rightarrow FG$ be a natural transformation. By Proposition 3.6, there exists a unique $\sum_{i=1}^n m_i \otimes_R f_i \in (M \otimes_R {}^*M)^S$ such that ξ_Q is given by (20). Then

$$(m \otimes_R f)(\xi_Q \circ \varepsilon_Q) = ((m)f)\xi_Q = \sum_{i=1}^n m_i \otimes_R (?)f_i \cdot (m)f,$$

hence G is a naturally full if and only if (22) holds, for every $Q \in {}_S\mathcal{M}$, $f \in {}_S\text{Hom}(M, Q)$ and $m \in M$. By Lemma 3.7, this is equivalent to (23).

2. Let $\nu : GF \rightarrow {}_1{}_R\mathcal{M}$ be a natural transformation; according to Proposition 3.6, there exists a unique $E \in {}_R\text{Hom}_R({}_S\text{End}(M), R)$ such that ν_P is given by (21). Then we easily compute that

$$\left(\left(\sum_i g_i \otimes_R p_i \right) (\psi_{M,P}) \right) (\eta_P \circ \nu_P) = \left(\sum_i (g_i)E \cdot p_i \right) \eta_P = \left(\sum_i g_i \otimes_R p_i \right) (\psi_{M,P})$$

if and only if

$$(m)\left(\sum_i (g_i)E \cdot p_i\right)\eta_P = (m)\left(\left(\sum_i g_i \otimes_R p_i\right)(\psi_{M,P})\right), \text{ for every } m \in M$$

if and only if

$$(m) \otimes_R \left(\sum_i (g_i)E \cdot p_i\right) = \sum_i (m)g_i \otimes_R p_i, \text{ for every } m \in M$$

if and only if

$$\sum_i m(g_i)E \otimes_R p_i = \sum_i (m)g_i \otimes_R p_i, \text{ for every } m \in M.$$

Therefore F is naturally full if and only if

$$\eta_P \circ \nu_P = GFP \text{ for every } P \in {}_R\mathcal{M}$$

if and only if

$$\sum_i m(g_i)E \otimes_R p_i = \sum_i (m)g_i \otimes_R p_i,$$

for all $P \in {}_R\mathcal{M}$, $p_i \in P$, $g_i \in {}_S\text{End}(M)$ and $m \in M$, if and only if

$$(m)f = mE(f),$$

for all $m \in M$ and $f \in {}_S\text{End}(M)$, if and only if

$$\chi \circ E = {}_S\text{End}(M).$$

This proves the equivalence of (1) and (2). The proof of the equivalence of (2) and (3) is identical to the proof of the equivalence of (2) and (3) in Proposition 3.1. \square

Remark 3.9. Proposition 3.1 is a special case of Theorem 3.8: let $i : R \rightarrow S$ be a ring homomorphism, and view $M = S$ as an (S, R) -bimodule.

Proposition 3.10. *Let $M \in {}_S\mathcal{M}_R$ and assume that the coinduction functor $G = {}_S\text{Hom}(M, \bullet)$ is naturally full. Let $\sum_{i=1}^n m_i \otimes_R f_i \in (M \otimes_R {}^*M)^S$ be as in Theorem 3.8. Then $e = \sum_{i=1}^n (m_i)f_i$ is a central idempotent of S and*

$$sm = sem = esem,$$

for all $s \in S$ and $m \in M$. Therefore $S \simeq S_1 \times S_2$ where $S_1 = eSe$, S acts on M via S_1 and M is a generator in ${}_{S_1}\mathcal{M}$.

Proof. Let $s \in S$. Since $\sum_{i=1}^n m_i \otimes_R f_i \in (M \otimes_R {}^*M)^S$, we have

$$\sum_{i=1}^n sm_i \otimes_R f_i = \sum_{i=1}^n m_i \otimes_R f_i \cdot s.$$

Applying ε_S to both sides, we find

$$\sum_{i=1}^n (sm_i)f_i = \sum_{i=1}^n (m_i)(f_i \cdot s).$$

Recall that the right S -action on *M is given by $(m)(f \cdot s) := (m)f \cdot s$; using the fact that f_i is left A -linear, we find that

$$\sum_{i=1}^n s \cdot (m_i) f_i = \sum_{i=1}^n ((m_i) f_i) \cdot s,$$

and this shows that $e = \sum_{i=1}^n (m_i) f_i$ is a central element of S . e is also idempotent since

$$e \cdot e = e \cdot (1_S) \xi_S \varepsilon_S = (e) \xi_S \varepsilon_S = (1_S) \xi_S \varepsilon_S \xi_S \varepsilon_S = (1_S) \xi_S \varepsilon_S = e,$$

where ξ_Q is given by (20).

Applying ε_M to both sides of (23), we find

$$m = \left(\sum_{i=1}^n (m_i) f_i \right) m = e \cdot m,$$

hence we have, for all $s \in S$ and $m \in M$:

$$sm = sem = esem.$$

Taking $Q = S$ in (22), we find

$$\mathrm{Tr}_M(S) := \sum_{f \in {}_S\mathrm{Hom}(M, S)} \mathfrak{I}(f) = \sum_{f \in {}_S\mathrm{Hom}(M, S)} \sum_{m \in M} (m) f = \mathfrak{I}(\varepsilon_S).$$

Now from $\xi_S \circ \varepsilon_S = M \otimes_R {}^*M$, it follows that

$$\mathfrak{I}(\varepsilon_S) = \mathfrak{I}(\varepsilon_S \circ \xi_S) = (S) \xi_S \varepsilon_S = Se = eSe.$$

□

Proposition 3.11. *Let $M \in {}_S\mathcal{M}_R$. If M is a generator of ${}_S\mathcal{M}$ and $\chi : R \rightarrow {}_S\mathrm{End}(M)$ is a ring epimorphism, then the coinduction functor $G = {}_S\mathrm{Hom}(M, \bullet)$ is fully faithful, that is, all counit maps ε_Q are isomorphisms.*

Proof. We will first show that G is naturally full. M is a generator of ${}_S\mathcal{M}$, so there exist $m_1, \dots, m_n \in M$ and $f_1, \dots, f_n \in {}^*M$ such that

$$1_S = \sum_{i=1}^n (m_i) f_i.$$

χ is a ring epimorphism, so it follows from Theorem 1.1 that

$$\varepsilon_M : M \otimes_R {}_S\mathrm{End}(M) \rightarrow M, \quad \varepsilon_M(m \otimes_R f) = m \cdot f = (m)f$$

is an isomorphism. From

$$\varepsilon_M(m \otimes_R M) = m = 1_S m = \sum_{i=1}^n (m_i) f_i m = \varepsilon_M\left(\sum_{i=1}^n m_i \otimes_R (?) f_i m\right),$$

we deduce that

$$m \otimes_R M = \sum_{i=1}^n m_i \otimes_R (?) f_i m,$$

for every $m \in M$. Hence $m_1, \dots, m_n \in M$ and f_1, \dots, f_n satisfy (23), and (22) holds, by Lemma 3.7. Now let us prove that $\sum_{i=1}^n m_i \otimes_R f_i \in (M \otimes_R {}^*M)^S$, or

$$(24) \quad \sum_{i=1}^n sm_i \otimes_R f_i = \sum_{i=1}^n m_i \otimes_R f_i s,$$

for every $s \in S$. We compute

$$\begin{aligned} \sum_{i=1}^n sm_i \otimes_R f_i &\stackrel{(22)}{=} \sum_{i,j=1}^n m_j \otimes_R (?)f_j ((sm_i)f_i) \\ &= \sum_{j=1}^n m_j \otimes_R \sum_{i=1}^n (?)f_j ((sm_i)f_i) = \sum_{j=1}^n m_j \otimes_R (?)f_j s. \end{aligned}$$

In fact, for every $m \in M$, we have:

$$\begin{aligned} (m) \sum_{i=1}^n (?)f_j ((sm_i)f_i) &= \sum_{i=1}^n (m)f_j ((sm_i)f_i) \\ &= (m)f_j s \sum_{i=1}^n ((m_i)f_i) = (m)f_j s = (m)((?)f_j s), \end{aligned}$$

and it follows from Theorem 3.8 that G is naturally full, which implies that all the counit maps ε_Q split in ${}_S\mathcal{M}$. Since ${}_SM$ is a generator, the counit map $\varepsilon_Q : M \otimes_R {}_S\text{Hom}(M, Q) \rightarrow Q$ is an epimorphism in ${}_S\mathcal{M}$, and therefore it is an isomorphism. \square

3.3. Corings. Let R be a ring. Recall that an R -coring is a comonoid in the monoidal category ${}_R\mathcal{M}_R$. Thus a coring is a triple $(\mathcal{C}, \Delta_{\mathcal{C}}, \varepsilon_{\mathcal{C}})$, where \mathcal{C} is an R -bimodule, and $\Delta : \mathcal{C} \rightarrow \mathcal{C} \otimes_R \mathcal{C}$ and $\varepsilon_{\mathcal{C}} : \mathcal{C} \rightarrow R$ are R -bimodule maps such that

$$(\Delta_{\mathcal{C}} \otimes_R \mathcal{C}) \circ \Delta_{\mathcal{C}} = (\mathcal{C} \otimes_R \Delta_{\mathcal{C}}) \circ \Delta_{\mathcal{C}}$$

and

$$(\varepsilon_{\mathcal{C}} \otimes_R \mathcal{C}) \circ \Delta_{\mathcal{C}} = (\mathcal{C} \otimes_R \varepsilon_{\mathcal{C}}) \circ \Delta_{\mathcal{C}} = \mathcal{C}.$$

$\Delta_{\mathcal{C}}$ is called the comultiplication, and $\varepsilon_{\mathcal{C}}$ is called the counit. We use the Sweedler-Heyneman notation

$$\Delta_{\mathcal{C}} = c_{(1)} \otimes_R c_{(2)},$$

where the summation is implicitly understood. A right \mathcal{C} -comodule is a couple (M, ρ^M) , where M is a right R -module, and $\rho^M : M \rightarrow M \otimes_R \mathcal{C}$ is a right R -linear map, called the coaction, satisfying the conditions

$$(\rho^M \otimes_R \mathcal{C}) \circ \rho^M = (M \otimes_R \Delta_{\mathcal{C}}) \circ \rho^M, \quad (M \otimes_R \varepsilon_{\mathcal{C}}) \circ \rho^M = M.$$

We use the following Sweedler-Heyneman notation for coactions:

$$\rho^M(m) = m_{[0]} \otimes_R m_{[1]}$$

Let M and N be right \mathcal{C} -comodules. A right R -linear map $f : M \rightarrow N$ is called right \mathcal{C} -colinear if

$$\rho^N(f(m)) = f(m_{[0]}) \otimes_R m_{[1]},$$

for all $m \in M$. The category of right \mathcal{C} -comodules and right \mathcal{C} -colinear maps is denoted by $\mathcal{M}_R^{\mathcal{C}}$.

Corings were introduced by Sweedler in [15], and recently revived by Brzeziński [1]. For a detailed study of corings, we refer to [3].

The functor $F : \mathcal{M}_R^{\mathcal{C}} \rightarrow \mathcal{M}_R$ forgetting the right \mathcal{C} -coaction has a right adjoint $G = \bullet \otimes_R \mathcal{C} : \mathcal{M}_R \rightarrow \mathcal{M}_R^{\mathcal{C}}$. For $N \in \mathcal{M}_R$, the \mathcal{C} -comodule structure on $G(M) = M \otimes_R \mathcal{C}$ is given by

$$(n \otimes_R c)r = n \otimes_R cr \text{ and } \rho^{G(M)}(n \otimes_R c) = n \otimes_R c_{(1)} \otimes_R c_{(2)}$$

The unit and counit of the adjunction are as follows, for $M \in \mathcal{M}_R^{\mathcal{C}}$ and $N \in \mathcal{M}_R$:

$$\begin{aligned} \eta_M &= \rho^M : M \rightarrow M \otimes_R \mathcal{C} \\ \varepsilon_N &= I_N \otimes_R \varepsilon_{\mathcal{C}} : N \otimes_R \mathcal{C} \rightarrow N \end{aligned}$$

We will now investigate when F and G are naturally full. In order to apply Theorem 2.6, we need to know $\underline{\text{Nat}}(1_{\mathcal{M}_R}, FG)$ and $\underline{\text{Nat}}(GF, 1_{\mathcal{M}^{\mathcal{C}}})$. This computation has been done in [8, Proposition 66 and 67]. The result is stated in the next Proposition.

Proposition 3.12. *Let \mathcal{C} be an R -coring, and consider the adjoint pair (F, G) introduced above. Then*

$$\underline{\text{Nat}}(1_{\mathcal{M}_R}, FG) \simeq \mathcal{C}^R = \{z \in \mathcal{C} \mid rz = zr, \text{ for all } r \in R\}.$$

The natural transformation ξ corresponding to $z \in \mathcal{C}^R$ is the following:

$$\xi_N : N \rightarrow N \otimes_R \mathcal{C}, \quad \xi_N(n) = n \otimes_R z.$$

Also

$$\begin{aligned} \underline{\text{Nat}}(GF, 1_{\mathcal{M}_R^{\mathcal{C}}}) &\simeq \{\vartheta \in {}_R\text{Hom}_R(\mathcal{C} \otimes_R \mathcal{C}, R) \\ &\mid c_{(1)}\vartheta(c_{(2)} \otimes_R d) = \vartheta(c \otimes_R d_{(1)})d_{(2)} \text{ for all } c, d \in \mathcal{C}\} \end{aligned}$$

The natural transformation ν corresponding to ϑ is the following:

$$\nu_M : M \otimes_R \mathcal{C} \rightarrow M, \quad \nu_M(m \otimes_R c) = m_{[0]}\vartheta(m_{[1]} \otimes_R c).$$

Proposition 3.13. *Let \mathcal{C} be an R -coring. With notation as above, we have:*

1. *The following statements are equivalent:*

- (1) *The functor $G = \bullet \otimes_R \mathcal{C} : \mathcal{M}_R \rightarrow \mathcal{M}_R^{\mathcal{C}}$ is naturally full;*
- (2) *there exists $z \in \mathcal{C}^R$ such that $c = \varepsilon_{\mathcal{C}}(c)z$, for all $c \in \mathcal{C}$;*
- (3) *$\varepsilon_{\mathcal{C}}$ splits in ${}_R\mathcal{M}_R$, i.e. there is $\xi : R \rightarrow \mathcal{C}$ in ${}_R\mathcal{M}_R$ such that $\xi \circ \varepsilon_{\mathcal{C}} = \text{id}_{\mathcal{C}}$.*

2. *The following statements are equivalent:*

- (1) *The functor $F : \mathcal{M}_R^{\mathcal{C}} \rightarrow \mathcal{M}_R$ is naturally full;*
- (2) *$c\varepsilon_{\mathcal{C}}(d) = \varepsilon_{\mathcal{C}}(c)d$, for all $c, d \in \mathcal{C}$;*

(3) $\Delta_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{C} \otimes_R \mathcal{C}$ is surjective (hence an isomorphism in ${}^{\mathcal{C}}_R \mathcal{M}_R^{\mathcal{C}}$).

Proof. 1. (1) \iff (2). Take a natural transformation $\xi : 1_{\mathcal{M}_R} \rightarrow FG$, and the corresponding $z \in \mathcal{C}^R$, as in Proposition 3.12. Then for all $N \in \mathcal{M}_R$, $n \in N$ and $c \in \mathcal{C}$, we have that

$$(\xi_N \circ \varepsilon_N)(n \otimes_R c) = \xi_N(n \varepsilon_{\mathcal{C}}(c)) = n \otimes_R \varepsilon_{\mathcal{C}}(c)z,$$

hence $\xi_N \circ \varepsilon_N = N \otimes_R \mathcal{C}$ if and only if $n \otimes_R c = n \otimes_R \varepsilon_{\mathcal{C}}(c)z$, for all $n \in N$ and $c \in \mathcal{C}$.

If G is naturally full, then there exists ξ such that $\xi_N \circ \varepsilon_N = N \otimes_R \mathcal{C}$, for all N . Taking $N = R$ and $n = 1$, we find that $c = \varepsilon_{\mathcal{C}}(c)z$, as needed. The converse is obvious.

(2) \implies (3). Let $\xi : R \rightarrow \mathcal{C}$ be defined by $\xi(r) := rz$.

(3) \implies (2). Set $z := \xi(1_R)$.

2. (1) \implies (3). By (13) of Theorem 2.6, $\eta_{\mathcal{C}}$ is surjective. Since $\Delta_{\mathcal{C}} = \eta_{\mathcal{C}}$, we conclude.

(3) \implies (2). Since $(\varepsilon_{\mathcal{C}} \otimes_R \mathcal{C}) \circ \Delta_{\mathcal{C}} = \mathcal{C} = (\mathcal{C} \otimes_R \varepsilon_{\mathcal{C}}) \circ \Delta_{\mathcal{C}}$ this implies $(\varepsilon_{\mathcal{C}} \otimes_R \mathcal{C}) = (\mathcal{C} \otimes_R \varepsilon_{\mathcal{C}})$ and hence $\varepsilon_{\mathcal{C}}(c)c' = c\varepsilon_{\mathcal{C}}(c')$ for any $c, c' \in \mathcal{C}$.

(2) \implies (1). Assume that $c\varepsilon_{\mathcal{C}}(d) = \varepsilon_{\mathcal{C}}(c)d$, for all $c, d \in \mathcal{C}$, and define $\vartheta : \mathcal{C} \otimes_R \mathcal{C} \rightarrow R$ by

$$\vartheta(c \otimes_R d) = \varepsilon_{\mathcal{C}}(c)\varepsilon_{\mathcal{C}}(d).$$

Then

$$c_{(1)}\vartheta(c_{(2)} \otimes_R d) = c\varepsilon_{\mathcal{C}}(d) = \varepsilon_{\mathcal{C}}(c)d = \vartheta(c \otimes_R d_{(1)})d_{(2)}.$$

The natural transformation ν corresponding to ϑ is then a right inverse of η , since

$$(\eta_M \circ \nu_M)(m \otimes_R c) = m_{[0]}\varepsilon_{\mathcal{C}}(m_{[1]}\varepsilon_{\mathcal{C}}(c_{(1)})) \otimes_R c_{(2)} = m \otimes_R c,$$

and it follows that F is naturally full. \square

Corollary 3.14. *Let \mathcal{C} be an R -bimodule. Then there is a bijection between the following sets:*

- (1) *the set of R -coring structures on \mathcal{C} such that the functor $G = \bullet \otimes_R \mathcal{C} : \mathcal{M}_R \rightarrow \mathcal{M}_R^{\mathcal{C}}$ is naturally full;*
- (2) *the set of R -ring structures on \mathcal{C} such that the restriction of scalars functor is naturally full.*

Proof. Let \mathcal{C} be an R -coring and assume that the functor $G = \bullet \otimes_R \mathcal{C} : \mathcal{M}_R \rightarrow \mathcal{M}_R^{\mathcal{C}}$ is naturally full. By Proposition 3.13, $\varepsilon_{\mathcal{C}}$ splits in ${}_R \mathcal{M}_R$, i.e. there is a $\xi : R \rightarrow \mathcal{C}$ in ${}_R \mathcal{M}_R$ such that $\xi \circ \varepsilon_{\mathcal{C}} = \mathcal{C}$. Then we have the following ring structure on \mathcal{C} :

$$c \cdot c' = \xi[\varepsilon_{\mathcal{C}}(c)\varepsilon_{\mathcal{C}}(c')] ; 1_{\mathcal{C}} = \xi(1_R).$$

Moreover, ξ becomes a ring morphism with an R -bimodule section, so the restriction of scalars functor is naturally full, by Proposition 3.1.

Conversely, assume that \mathcal{C} is an R -ring, such that the restriction of scalars

functor is naturally full. Then there is a ring morphism $\varphi : R \rightarrow \mathcal{C}$ with a section $E \in {}_R\text{Hom}_R(\mathcal{C}, R)$. We define an R -coring structure on \mathcal{C} as follows:

$$\Delta(c) = c \otimes_R 1_{\mathcal{C}} ; \varepsilon_{\mathcal{C}}(c) = E(c).$$

It is straightforward to check that the two constructions are inverse to each other. For example, take an R -coring structure on \mathcal{C} , and consider the associated R -ring structure. Then

$$c \otimes_R 1_{\mathcal{C}} = \sum c_{(1)} \otimes_R \varepsilon_{\mathcal{C}}(c_{(2)}) 1_{\mathcal{C}} = \sum c_{(1)} \otimes_R \xi(\varepsilon_{\mathcal{C}}(c_{(2)})) 1_{\mathcal{C}} = \sum c_{(1)} \otimes_R c_{(2)},$$

as needed. \square

Corollary 3.15. *Let \mathcal{C} be an R -coring. 1. Assume that $G = \bullet \otimes_R \mathcal{C} : \mathcal{M}_R \rightarrow \mathcal{M}_R^{\mathcal{C}}$ is naturally full. Then we have the following properties.*

- (1) *The functor $F : \mathcal{M}_R^{\mathcal{C}} \rightarrow \mathcal{M}_R$ is naturally full; the converse property holds if there exists a $z \in \mathcal{C}$ such that $\varepsilon_{\mathcal{C}}(z) = 1_R$;*
- (2) *\mathcal{C} is finitely generated and projective as a left (right) R -module;*
- (3) *the functor $F : \mathcal{M}_R^{\mathcal{C}} \rightarrow \mathcal{M}_R$ is a Frobenius functor.*

2. *If $F : \mathcal{M}_R^{\mathcal{C}} \rightarrow \mathcal{M}_R$ is naturally full, then it is also separable.*

Proof. 1. If G is naturally full, then there exists $z \in \mathcal{C}^R$ such that $c = \varepsilon_{\mathcal{C}}(c)z$, for all $c \in \mathcal{C}$.

(1) We easily compute that

$$c\varepsilon_{\mathcal{C}}(d) = \varepsilon_{\mathcal{C}}(c)z\varepsilon_{\mathcal{C}}(d) = \varepsilon_{\mathcal{C}}(c)\varepsilon_{\mathcal{C}}(d)z = \varepsilon_{\mathcal{C}}(c)d,$$

for all $c, d \in \mathcal{C}$, and it follows from Proposition 3.13 that F is naturally full. Conversely, assume that F is naturally full, and that there exists $z \in \mathcal{C}$ such that $\varepsilon_{\mathcal{C}}(z) = 1_R$. Then $c\varepsilon_{\mathcal{C}}(d) = \varepsilon_{\mathcal{C}}(c)d$, for all $c, d \in \mathcal{C}$, and, in particular $c = c\varepsilon_{\mathcal{C}}(z) = \varepsilon_{\mathcal{C}}(c)z$. $z \in \mathcal{C}^R$ since $zr = \varepsilon_{\mathcal{C}}(zr)z = \varepsilon_{\mathcal{C}}(z)rz = rz$, for all $r \in R$, and it follows from Proposition 3.13 that G is naturally full.

(2) By Proposition 3.13, the functor $G = \bullet \otimes_R \mathcal{C} : \mathcal{M}_R \rightarrow \mathcal{M}_R^{\mathcal{C}}$ is naturally full if and only if there is $\xi : R \rightarrow \mathcal{C}$ in ${}_R\mathcal{M}_R$ such that $\xi \circ \varepsilon_{\mathcal{C}} = \mathcal{C}$. Thus \mathcal{C} is finitely generated and projective both as a right and as a left R -module.

(3) It is known (see [1, Theorem 4.1]) that the following statements are equivalent:

- The forgetful functor $F : \mathcal{M}_R^{\mathcal{C}} \rightarrow \mathcal{M}_R$ is Frobenius;
- \mathcal{C} is a finitely generated left R -module and there exists $e \in \mathcal{C}^R$ such that the map

$$\phi : {}_R\text{Hom}(\mathcal{C}, R) \rightarrow \mathcal{C}, \phi(f) = \sum e_{(1)}f(e_{(2)})$$

is bijective.

Put $e = z$ and observe that

$$\sum z_{(1)} \otimes_R z_{(2)} = \sum z_{(1)} \otimes_R \varepsilon_{\mathcal{C}}(z_{(2)})z = z \otimes_R z,$$

so that $\phi(f) = zf(z)$. Consider the map

$$v : \mathcal{C} \rightarrow {}_R\text{Hom}(\mathcal{C}, R), (v(c))(c') = \varepsilon_{\mathcal{C}}(c')\varepsilon_{\mathcal{C}}(c).$$

We then compute that

$$(\phi \circ v)(c) = z((v)(c))(z) = z\varepsilon_{\mathcal{C}}(z)\varepsilon_{\mathcal{C}}(c) = z\varepsilon_{\mathcal{C}}(z\varepsilon_{\mathcal{C}}(c)) = z\varepsilon_{\mathcal{C}}(c) = c,$$

and

$$\begin{aligned} ((v \circ \phi)(f))(c) &= ((v(zf(z)))(c) = \varepsilon_{\mathcal{C}}(c)\varepsilon_{\mathcal{C}}(zf(z)) \\ &= \varepsilon_{\mathcal{C}}(\varepsilon_{\mathcal{C}}(c)z)f(z) = \varepsilon_{\mathcal{C}}(c)f(z) = f(\varepsilon_{\mathcal{C}}(c)z) = f(c). \end{aligned}$$

So ϕ is bijective and we know from (2) that \mathcal{C} is finitely generated and projective.

2. If $F : \mathcal{M}^{\mathcal{C}} \rightarrow \mathcal{M}_R$ is naturally full, $\Delta_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{C} \otimes_R \mathcal{C}$ is an isomorphism in ${}^{\mathcal{C}}\mathcal{M}_R^{\mathcal{C}}$ (the inverse is $\varepsilon_{\mathcal{C}} \otimes_R \mathcal{C} = \mathcal{C} \otimes_R \varepsilon_{\mathcal{C}}$). In particular, \mathcal{C} is coseparable and hence, by [1, Corollary 3.6], F is separable. \square

Examples 3.16. 1. Let $\varphi : R \rightarrow S$ be a ring extension. Then $\mathcal{C} = S \otimes_R S$ is an S -coring with comultiplication $\Delta_{\mathcal{C}} : S \otimes_R S \rightarrow (S \otimes_R S) \otimes_S (S \otimes_R S)$ and counit $\varepsilon_{\mathcal{C}} : S \otimes_R S \rightarrow S$ given by

$$\Delta_{\mathcal{C}}(a \otimes_R a') = (a \otimes_R 1_S) \otimes_S (1_S \otimes_R a') \text{ and } \varepsilon_{\mathcal{C}}(a \otimes_R a') = aa'.$$

\mathcal{C} is called the Sweedler coring associated to the extension $\varphi : R \rightarrow S$. The functor F is naturally full if and only if the functor G is full if and only if φ is a ring epimorphism.

2. Let I be a two-sided idempotent ideal of a ring R and assume that I is a pure right R -submodule. Then the multiplication map $m_I : I \otimes_R I \rightarrow I$ is bijective and I is an R -coring where $\Delta = m^{-1} : I \rightarrow I \otimes_R I$ and $\varepsilon : I \rightarrow R$ is the inclusion map. It follows from Proposition 3.13 that F is naturally full. By Proposition 3.13, if G is naturally full then I must be finitely generated and projective as a left (right) R -module, but this is not true in general.

Remark 3.17. Let R and S be rings and $M \in {}_S\mathcal{M}_R$ an (S, R) -bimodule. Then ${}^*M = {}_S\text{Hom}(M, S) \in {}_R\mathcal{M}_S$. Recall (see Section 3.2) that we have a pair of adjoint functors

$$F = M \otimes_R \bullet : {}_R\mathcal{M} \rightarrow {}_S\mathcal{M} ; G = {}_S\text{Hom}(M, \bullet) : {}_S\mathcal{M} \rightarrow {}_R\mathcal{M}.$$

Now assume that ${}_S M$ is finitely generated and projective, and let $\{(e_i, {}^*e_i) \mid i = 1, \dots, n\}$ be a finite dual basis for M . Then we have a coring structure on the S -bimodule $\mathcal{C} = M \otimes_R {}^*M$, given by

$$\Delta(m \otimes_R \mu) = \sum_i m \otimes_R {}^*e_i \otimes_S e_i \otimes_R \mu \text{ and } \varepsilon(m \otimes_R \mu) = (m)\mu.$$

\mathcal{C} is called the comatrix coring associated to M . We refer to [2, 5, 10] for a detailed study of comatrix corings. We can consider the adjoint pair discussed in Proposition 3.12, in the right handed case:

$$F_{\mathcal{C}} : {}^{\mathcal{C}}\mathcal{M} \rightarrow {}_S\mathcal{M} ; G_{\mathcal{C}} = \mathcal{C} \otimes_S \bullet : {}_S\mathcal{M} \rightarrow {}^{\mathcal{C}}\mathcal{M},$$

where $F_{\mathcal{C}}$ forgets the left \mathcal{C} -coaction. Since M is finitely generated and projective as a left S -module, we have for any left S -module that

$$F_{\mathcal{C}}G_{\mathcal{C}}(N) = M \otimes_R {}^*M \otimes_S N \cong M \otimes_R {}_S\text{Hom}(M, N) = FG(N).$$

It is easy check that this isomorphism of left S -modules is natural in N and that it preserves the counits of the two adjunctions. Thus we obtain:

- (1) $\underline{\text{Nat}}(1_{S\mathcal{M}}, FG) \simeq \underline{\text{Nat}}(1_{S\mathcal{M}}, F_{\mathcal{C}}G_{\mathcal{C}})$;
- (2) G is naturally full (resp. separable) if and only if $G_{\mathcal{C}}$ is naturally full (resp. separable).

This explains the similarity between the first parts of Propositions 3.6 and 3.12. Moreover it tells that G is naturally full if and only if there exists $z \in (M \otimes_R {}^*M)^S$ such that $m \otimes_R \mu = z \cdot (m)\mu$, for all $m \otimes_R \mu \in M \otimes_R {}^*M$ (apply the left sided version of Proposition 3.13).

Now let us look at the R -bimodule $A = {}^*M \otimes_S M \cong {}_S\text{End}(M)$. Thus A is a ring, with multiplication and counit given by

$$(\mu \otimes_S m) \cdot (\tau \otimes_S t) = \mu \otimes_S (m)\tau \cdot t ; \quad 1_A = \sum_i {}^*e_i \otimes_S e_i,$$

and we have a ring morphism

$$\varphi : R \rightarrow A, \quad \varphi(r) = 1_A \cdot r = r \cdot 1_A.$$

We then have a pair of adjoint functors

$$F_A = A \otimes_R \bullet : {}_R\mathcal{M} \rightarrow {}_A\mathcal{M} ; \quad G_A : {}_A\mathcal{M} \rightarrow {}_R\mathcal{M},$$

where G_A is restriction of scalars. Since M is finitely generated and projective as a left S -module, we have, for any $N \in {}_R\mathcal{M}$,

$$G_A F_A(N) = {}^*M \otimes_S M \otimes_R N \cong {}_S\text{Hom}(M, M \otimes_R N) = GF(N).$$

This isomorphism is natural in N and preserves the units of the adjunctions. Hence we obtain the following:

- (1) $\underline{\text{Nat}}(GF, 1_{{}_R\mathcal{M}}) \simeq \underline{\text{Nat}}(F_A G_A, 1_{{}_A\mathcal{M}})$;
- (2) F is naturally full (resp. separable) if and only if F_A is naturally full (resp. separable).

This explains the similarity between the second parts of Propositions 3.6 and 3.12. Moreover it clarifies the analogy between the second parts of Proposition 3.1 and Theorem 3.8.

Let \mathcal{C} be an R -coring. $g \in \mathcal{C}$ is called grouplike if $\Delta_{\mathcal{C}}(g) = g \otimes_R g$ and $\varepsilon_{\mathcal{C}}(g) = 1$. There is a bijective correspondence between the grouplike elements of \mathcal{C} and right (or left) \mathcal{C} -comodule structures on R , see [1]: the right \mathcal{C} -coaction on R corresponding to g is given by $\rho(r) = 1 \otimes_R gr$.

Fix a grouplike element $g \in \mathcal{C}$, and define the coinvariants $M^{\text{co}\mathcal{C}}$ of M by

$$M^{\text{co}\mathcal{C}} = \{m \in M \mid \rho_M(m) = m \otimes_R g\}.$$

In particular,

$$B = R^{\text{co}\mathcal{C}} = \{r \in R \mid rg = gr\}$$

is a subring of R and $M^{\text{co}\mathcal{C}} \in \mathcal{M}_B$. We obtain a functor $\tilde{G} = (\bullet)^{\text{co}\mathcal{C}} : \mathcal{M}_R^{\mathcal{C}} \rightarrow \mathcal{M}_B$, which has a left adjoint $\tilde{F} = \bullet \otimes_B R$. For any $N \in \mathcal{M}_B$, $N \otimes_B R$ is a

right \mathcal{C} -comodule via the right \mathcal{C} -coaction on R . The unit and counit of the adjunction are the following, for every $N \in \mathcal{M}_B$ and $M \in \mathcal{M}_R^{\mathcal{C}}$:

$$\alpha_N : N \rightarrow (N \otimes_B R)^{\text{co}\mathcal{C}}, \quad \alpha_N(n) = n \otimes_B 1;$$

$$\beta_M : M^{\text{co}\mathcal{C}} \otimes_B R \rightarrow M, \quad \beta_M(m \otimes_B a) = ma.$$

Proposition 3.18. *Let \mathcal{C} be an R -coring with a fixed grouplike element g . If there exists an (R, R) -bimodule map $\chi : \mathcal{C} \otimes_R \mathcal{C} \rightarrow R$ such that*

$$(25) \quad c_{(1)}\chi(c_{(2)} \otimes_R d) = \chi(c \otimes_R d_{(1)})d_{(2)},$$

for all $c, d \in \mathcal{C}$ and

$$\chi(g \otimes_R g) = 1,$$

then the functor $\tilde{F} = \bullet \otimes_B R : \mathcal{M}_B \rightarrow \mathcal{M}_R^{\mathcal{C}}$ is fully faithful, which means that it is separable and naturally full.

Proof. We first show that the map

$$t : R \rightarrow B, \quad t(r) = \chi(gr \otimes_R g)$$

is a B -bimodule map. t is left B -linear since

$$t(br) = \chi(gbr \otimes_R g) = \chi(bgr \otimes_R g) = b\chi(gr \otimes_R g) = bt(r),$$

for all $b \in B$ and $r \in R$. A similar argument shows that t is right B -linear. Let us show that $t(r) \in B$, for all $r \in R$: applying (25) with $c = gr$, $d = g$, we find that $g\chi(gr \otimes_R g) = \chi(gr \otimes_R g)g$, hence $t(r) = \chi(gr \otimes_R g) \in B$. Also observe that $t^2 = t$, since $t(b) = b$, for $b \in B$.

Take $N \in \mathcal{M}_B$. We claim that the map

$$\theta_N : (N \otimes_B R)^{\text{co}\mathcal{C}} \rightarrow N, \quad \theta_N\left(\sum_i n_i \otimes_B r_i\right) = \sum_i n_i t(r_i) = \sum_i n_i \chi(g \otimes_R r_i g)$$

is inverse to α_N . Obviously $(\theta_N \circ \alpha_N)(n) = n$, for all $n \in N$.

Take $\sum_i n_i \otimes_B r_i \in (N \otimes_B R)^{\text{co}\mathcal{C}}$. Then

$$\sum_i n_i \otimes_B gr_i = \sum_i n_i \otimes_B r_i g,$$

so

$$\sum_i n_i \otimes_B gr_i \otimes_R g = \sum_i n_i \otimes_B r_i g \otimes_R g,$$

and

$$\sum_i n_i \otimes_B \chi(gr_i \otimes_R g) = \sum_i n_i \otimes_B \chi(r_i g \otimes_R g).$$

Since $\chi(gr_i \otimes_R g) \in B$ and χ is left R -linear, we find

$$\sum_i n_i \chi(gr_i \otimes_R g) \otimes_B 1_R = \sum_i n_i \otimes_B r_i \chi(g \otimes_R g),$$

and, since $\chi(g \otimes_R g) = 1$,

$$(\alpha_N \circ \theta_N)\left(\sum_i n_i \otimes_B r_i\right) = \sum_i n_i \otimes_B r_i.$$

□

Recall from [1] that a coring \mathcal{C} is called coseparable if the forgetful functor $F : \mathcal{M}_R^{\mathcal{C}} \rightarrow \mathcal{M}^R$ is separable; this is equivalent to the existence of an (R, R) -bimodule map $\xi : \mathcal{C} \otimes_R \mathcal{C} \rightarrow R$ satisfying (25) such that $\chi \circ \Delta_{\mathcal{C}} = \varepsilon_{\mathcal{C}}$. If \mathcal{C} is coseparable, then the conditions of Proposition 3.18 are satisfied, and \tilde{F} is fully faithful, for every choice of the grouplike element g .

3.4. Homomorphisms of corings. Let \mathcal{C} be an R -coring and let \mathcal{D} be an S -coring. A coring homomorphism from \mathcal{C} to \mathcal{D} is a pair (Φ, φ) , where $\varphi : R \rightarrow S$ is a ring homomorphism and $\Phi : \mathcal{C} \rightarrow \mathcal{D}$ is a homomorphism of R -modules such that:

$$\omega_{\mathcal{D}, \mathcal{D}} \circ (\Phi \otimes_R \Phi) \circ \Delta_{\mathcal{C}} = \Delta_{\mathcal{D}} \circ \Phi ; \quad \varphi \circ \varepsilon_{\mathcal{C}} = \varepsilon_{\mathcal{D}} \circ \Phi,$$

where $\omega_{\mathcal{D}, \mathcal{D}} : \mathcal{D} \otimes_R \mathcal{D} \rightarrow \mathcal{D} \otimes_S \mathcal{D}$ is the canonical map induced by φ .

We have a functor $S \otimes_R \bullet : {}_{\mathcal{C}}^{\mathcal{C}}\mathcal{M} \rightarrow {}_{\mathcal{D}}^{\mathcal{D}}\mathcal{M}$, where the left \mathcal{D} -comodule structure of $S \otimes_R M$ is given by the map $s \otimes_R m \mapsto \sum s\Phi(m_{[-1]}) \otimes_S 1_S \otimes_R m_{[0]}$. Observe that $S \otimes_R \mathcal{C}$ is a right \mathcal{C} -comodule via $S \otimes_R \Delta_{\mathcal{C}}$. In a similar way, we have a functor $\bullet \otimes_R S : \mathcal{M}_R^{\mathcal{C}} \rightarrow \mathcal{M}_S^{\mathcal{D}}$.

Let L be an (S, R) -bimodule, and let $\rho_L^{\mathcal{D}} : L \rightarrow L \otimes_S \mathcal{D}$ and ${}^{\mathcal{C}}\rho_L : L \rightarrow \mathcal{C} \otimes_R L$ be comodule structures on L . It is clear that ${}^{\mathcal{C}}\rho_L$ is a morphism of right \mathcal{D} -comodules if and only if $\rho_L^{\mathcal{D}}$ is a morphism of left \mathcal{C} -comodules. In this case we say that L is a $(\mathcal{C}, \mathcal{D})$ -bicomodule; we can then define the category ${}^{\mathcal{C}}\mathcal{M}^{\mathcal{D}}$ of $(\mathcal{C}, \mathcal{D})$ -bicomodules. The morphisms are (R, S) -bimodule maps that are left \mathcal{C} -colinear and right \mathcal{D} -colinear.

Let $f, g : X \rightarrow Y$ be a pair of morphisms of right modules over a ring R and let $i : E \rightarrow X$ be its equalizer (that is the kernel of $f - g$). We will say that a left R -module ${}_R M$ preserves the equalizer of (f, g) if the map $i \otimes_R M : E \otimes_R M \rightarrow X \otimes_R M$ is the equalizer of the pair $(f \otimes_R M, g \otimes_R M)$. Analogously for the right-hand side.

Remark 3.19. Proposition 3.20 and the subsequent Theorems are inspired by [11]. The referee pointed out to us that it is not sufficient, as in the original statement in [11], to assume that ${}_R \mathcal{C}$ preserves equalizers. In fact, one needs the assumption that ${}_R(\mathcal{C} \otimes_R \mathcal{C})$ preserves equalizers. More precisely, let $M = N \otimes_S S \otimes_R \mathcal{C}$ and let (K, k) be the equalizer of $\rho_N^{\mathcal{D}} \otimes_S S \otimes_R \mathcal{C}$ and $N \otimes_S {}^{\mathcal{D}}\rho_{S \otimes_R \mathcal{C}}$. One has to prove that (K, k) inherits a \mathcal{C} -bicomodule structure. Now, by means of the universal property of the equalizers, there is a unique morphism $\rho_K^{\mathcal{C}} : K \rightarrow K \otimes_R \mathcal{C}$ such that

$$(k \otimes_R \mathcal{C})\rho_K^{\mathcal{C}} = \rho_M^{\mathcal{C}} k.$$

Using the coassociativity of the coaction of M and the relation above, we obtain:

$$\begin{aligned} (k \otimes_R \mathcal{C} \otimes_R \mathcal{C})(\rho_K^{\mathcal{C}} \otimes_R \mathcal{C})\rho_K^{\mathcal{C}} &= [(k \otimes_R \mathcal{C})\rho_K^{\mathcal{C}} \otimes_R \mathcal{C}]\rho_K^{\mathcal{C}} \\ &= (\rho_M^{\mathcal{C}} k \otimes_R \mathcal{C})\rho_K^{\mathcal{C}} = (\rho_M^{\mathcal{C}} \otimes_R \mathcal{C})(k \otimes_R \mathcal{C})\rho_K^{\mathcal{C}} \end{aligned}$$

$$\begin{aligned}
&= (\rho_M^{\mathcal{C}} \otimes_R \mathcal{C}) \rho_M^{\mathcal{C}} k = (M \otimes_R \Delta_{\mathcal{C}}) \rho_M^{\mathcal{C}} k \\
&= (M \otimes_R \Delta_{\mathcal{C}}) (k \otimes_R \mathcal{C}) \rho_K^{\mathcal{C}} = (k \otimes_R \mathcal{C} \otimes_R \mathcal{C}) (K \otimes_R \Delta_{\mathcal{C}}) \rho_K^{\mathcal{C}}.
\end{aligned}$$

Thus, since ${}_R(\mathcal{C} \otimes_R \mathcal{C})$ preserves equalizers, the morphism $k \otimes_R \mathcal{C} \otimes_R \mathcal{C}$ is a monomorphism so that we get

$$(\rho_K^{\mathcal{C}} \otimes_R \mathcal{C}) \rho_K^{\mathcal{C}} = (K \otimes_R \Delta_{\mathcal{C}}) \rho_K^{\mathcal{C}}.$$

Note also that ${}_R\mathcal{C}$ preserves equalizers, whenever ${}_R(\mathcal{C} \otimes_R \mathcal{C})$ does. Therefore this requirement is a little stronger and it has replaced the first one in all the following results.

Proposition 3.20. (cf. [11, Proposition 5.4]) *The (S, R) -bimodule $S \otimes_R \mathcal{C}$ is a $(\mathcal{D}, \mathcal{C})$ -bicomodule. Moreover, if ${}_R(\mathcal{C} \otimes_R \mathcal{C})$ preserves the equalizer of $(\rho_N^{\mathcal{D}} \otimes_S S \otimes_R \mathcal{C}, N \otimes_S {}^{\mathcal{D}}\rho_{S \otimes_R \mathcal{C}})$ for every right \mathcal{D} -comodule N , then the functor*

$$G = \bullet \square_{\mathcal{D}}(S \otimes_R \mathcal{C}) : \mathcal{M}_S^{\mathcal{D}} \rightarrow \mathcal{M}_R^{\mathcal{C}}$$

is right adjoint to

$$F = \bullet \otimes_R S : \mathcal{M}_R^{\mathcal{C}} \rightarrow \mathcal{M}_S^{\mathcal{D}}.$$

The unit $\eta : 1_{\mathcal{M}_R^{\mathcal{C}}} \rightarrow GF$ and the counit $\varepsilon : FG \rightarrow 1_{\mathcal{M}_S^{\mathcal{D}}}$ are given by

$$\eta_M(m) = \sum (m_{[0]} \otimes_R 1_S) \square_{\mathcal{D}}(1_S \otimes_R m_{[1]})$$

and

$$\varepsilon_N((n \square_{\mathcal{D}}(s \otimes_R c)) \otimes_R s') := ns\varphi(\varepsilon_{\mathcal{C}}(c))s'$$

for all $M \in \mathcal{M}_R^{\mathcal{C}}$ and $N \in \mathcal{M}_S^{\mathcal{D}}$.

Theorem 3.21. *Assume that ${}_R(\mathcal{C} \otimes_R \mathcal{C})$ preserves the equalizer of $(\rho_N^{\mathcal{D}} \otimes_S S \otimes_R \mathcal{C}, N \otimes_S {}^{\mathcal{D}}\rho_{S \otimes_R \mathcal{C}})$ for every $N \in \mathcal{M}_S^{\mathcal{D}}$, and that M_R preserves the equalizer of $(\rho_{\mathcal{C} \otimes_R S}^{\mathcal{D}} \otimes_S S \otimes_R \mathcal{C}, \mathcal{C} \otimes_R S \otimes_S {}^{\mathcal{D}}\rho_{S \otimes_R \mathcal{C}})$ for every $M \in \mathcal{M}_R^{\mathcal{C}}$. The functor $F = \bullet \otimes_R S : \mathcal{M}_R^{\mathcal{C}} \rightarrow \mathcal{M}_S^{\mathcal{D}}$ is naturally full if and only if $\eta_{\mathcal{C}} : \mathcal{C} \rightarrow GFC$ cosplits in ${}_R\mathcal{M}_R^{\mathcal{C}}$, i.e. there is a homomorphism of \mathcal{C} -bicomodules $\nu_{\mathcal{C}} : GFC \rightarrow \mathcal{C}$ such that $\eta_{\mathcal{C}} \circ \nu_{\mathcal{C}} = GFC$.*

Proof. By Theorem 2.6, F is naturally full if and only if $\eta : 1_{\mathcal{M}_R^{\mathcal{C}}} \rightarrow GF$ cosplits, i.e. there exists a natural transformation $\nu : GF \rightarrow 1_{\mathcal{M}_R^{\mathcal{C}}}$ such that $\eta_M \circ \nu_M = GFM$ for all $M \in \mathcal{M}_R^{\mathcal{C}}$. In this case, obviously $\eta_{\mathcal{C}} \circ \nu_{\mathcal{C}} = GFC$ and $\nu_{\mathcal{C}}$ is the required map.

Conversely, assume that there is a homomorphism of \mathcal{C} -bicomodules $\nu_{\mathcal{C}} : GFC \rightarrow \mathcal{C}$ such that $\eta_{\mathcal{C}} \circ \nu_{\mathcal{C}} = GFC$. Let $r_M : M \otimes_R R \rightarrow M$ be the canonical isomorphism. According to [11, Theorem 5.6], we have a natural transformation $\nu : GF \rightarrow 1_{\mathcal{M}_R^{\mathcal{C}}}$ defined as follows:

$$\nu_M = r_M \circ (M \otimes_R \varepsilon_{\mathcal{C}} \circ \nu_{\mathcal{C}}) \circ \kappa_M,$$

where $\kappa_M : GFM \rightarrow M \otimes_R GFC$ is a natural isomorphism (see the proof of [11, Theorem 5.6]). It is also proved in [11] that that $\kappa_M \circ \eta_M = (M \otimes_R \eta_{\mathcal{C}}) \circ \rho_M^{\mathcal{C}}$. In a similar way, we can show that

$$(\rho_M^{\mathcal{C}} \otimes_R GFC) \circ \kappa_M = (M \otimes_R \kappa_{\mathcal{C}}) \circ \kappa_M.$$

Hence we have:

$$\begin{aligned}
\rho_M^{\mathcal{C}} \circ \nu_M &= \rho_M^{\mathcal{C}} \circ r_M \circ (M \otimes_R \varepsilon_{\mathcal{C}} \circ \nu_{\mathcal{C}}) \circ \kappa_M \\
&= (M \otimes_R r_{\mathcal{C}}) \circ (\rho_M^{\mathcal{C}} \otimes_R R) \circ (M \otimes_R \varepsilon_{\mathcal{C}} \circ \nu_{\mathcal{C}}) \circ \kappa_M \\
&= (M \otimes_R r_{\mathcal{C}}) \circ (M \otimes_R \mathcal{C} \otimes_R \varepsilon_{\mathcal{C}} \circ \nu_{\mathcal{C}}) \circ (\rho_M^{\mathcal{C}} \otimes_R GFC) \circ \kappa_M \\
&= (M \otimes_R r_{\mathcal{C}}) \circ (M \otimes_R \mathcal{C} \otimes_R \varepsilon_{\mathcal{C}} \circ \nu_{\mathcal{C}}) \circ (M \otimes_R \kappa_{\mathcal{C}}) \circ \kappa_M \\
&= (M \otimes_R \nu_{\mathcal{C}}) \circ \kappa_M.
\end{aligned}$$

Now let us compute:

$$\begin{aligned}
\kappa_M \circ \eta_M \circ \nu_M &= (M \otimes_R \eta_{\mathcal{C}}) \circ \rho_M^{\mathcal{C}} \circ \nu_M \\
&= (M \otimes_R \eta_{\mathcal{C}}) \circ (M \otimes_R \nu_{\mathcal{C}}) \circ \kappa_M = \kappa_M.
\end{aligned}$$

κ_M is a monomorphism, so it follows that $\eta_M \circ \nu_M = GFM$. \square

Theorem 3.22. *Assume that ${}_R S$ and ${}_R \mathcal{C}$ preserve the equalizer of $(\rho_N^{\mathcal{D}} \otimes S \otimes_R \mathcal{C}, N \otimes_S {}^{\mathcal{D}} \rho_{S \otimes_R \mathcal{C}})$ for every $N \in \mathcal{M}_S^{\mathcal{D}}$. The functor $G = \bullet \square_{\mathcal{D}}(S \otimes_R \mathcal{C}) : \mathcal{M}_S^{\mathcal{D}} \rightarrow \mathcal{M}_R^{\mathcal{C}}$ is naturally full if and only if the \mathcal{D} -bicomodule map*

$$\hat{\Phi} : S \otimes_R \mathcal{C} \otimes_R S \rightarrow \mathcal{D}, \quad \hat{\Phi}(s \otimes_R c \otimes_R s) = s\Phi(c)s'$$

splits in ${}^{\mathcal{D}} \mathcal{M}_S^{\mathcal{D}}$, that is, there is a homomorphism of \mathcal{D} -bicomodules $\hat{\Psi} : \mathcal{D} \rightarrow S \otimes_R \mathcal{C} \otimes_R S$ such that $\hat{\Psi} \circ \hat{\Phi} = S \otimes_R \mathcal{C} \otimes_R S$.

Proof. By Theorem 2.6, G is naturally full if and only if $\varepsilon : FG \rightarrow 1_{\mathcal{M}_S^{\mathcal{D}}}$ splits, that is, there exists a natural transformation $\xi : 1_{\mathcal{M}_S^{\mathcal{D}}} \rightarrow FG$ such that $\xi_N \circ \varepsilon_N = FGN$ for all $N \in \mathcal{M}_S^{\mathcal{D}}$. In particular $\xi_{\mathcal{D}} \circ \varepsilon_{\mathcal{D}} = FGD$. In [11, 5.6], it is proved that

$$\hat{\Phi} = \varepsilon_{\mathcal{D}} \circ ({}^{\mathcal{D}} \rho_{S \otimes_R \mathcal{C}} \otimes_R S).$$

Let $l_N : S \otimes_S N \rightarrow N$ be the canonical isomorphism, for every $N \in \mathcal{M}_S^{\mathcal{D}}$. Then we can write

$$\begin{aligned}
&(l_{S \otimes_R \mathcal{C}} \otimes_R S) \circ (\varepsilon_{\mathcal{D}} \otimes_S (S \otimes_R \mathcal{C}) \otimes_R S) \circ \xi_{\mathcal{D}} \circ \hat{\Phi} \\
&= (l_{S \otimes_R \mathcal{C}} \otimes_R S) \circ (\varepsilon_{\mathcal{D}} \otimes_S (S \otimes_R \mathcal{C}) \otimes_R S) \circ ({}^{\mathcal{D}} \rho_{S \otimes_R \mathcal{C}} \otimes_R S) \\
&= (l_{S \otimes_R \mathcal{C}} \otimes_R S) \circ (l_{S \otimes_R \mathcal{C}}^{-1} \otimes_R S) = S \otimes_R \mathcal{C} \otimes_R S,
\end{aligned}$$

so we can choose $\hat{\Psi} = (l_{S \otimes_R \mathcal{C}} \otimes_R S) \circ (\varepsilon_{\mathcal{D}} \otimes_S (S \otimes_R \mathcal{C}) \otimes_R S) \circ \xi_{\mathcal{D}}$.

Conversely, assume that there is a homomorphism of \mathcal{D} -bicomodules $\hat{\Psi} : \mathcal{D} \rightarrow S \otimes_R \mathcal{C} \otimes_R S$ such that $\hat{\Psi} \circ \hat{\Phi} = S \otimes_R \mathcal{C} \otimes_R S$. In [11, Theorem 5.8], it is proved that the map

$$N \xrightarrow{\rho_N^{\mathcal{D}}} N \otimes_S \mathcal{D} \xrightarrow{N \otimes_S \hat{\Psi}} N \otimes_S S \otimes_R \mathcal{C} \otimes_R S$$

factorizes through a natural transformation

$$\xi_N : N \rightarrow FGN = (N \square_{\mathcal{D}}(S \otimes_R \mathcal{C})) \otimes_R S.$$

Now let $i : GN \rightarrow N \otimes_S (S \otimes_R \mathcal{C})$, for every $N \in \mathcal{M}_S^{\mathcal{D}}$, be the equalizer of $(\rho_N^{\mathcal{D}} \otimes_S S \otimes_R \mathcal{C}, N \otimes_S^{\mathcal{D}} \rho_{S \otimes_R \mathcal{C}})$ which is preserved by ${}_R S$. Then the morphism $i \otimes_R S$ is a monomorphism. In [11] we also find that:

$$(i \otimes_R S) \circ \xi_N = (N \otimes_S \hat{\Psi}) \circ \rho_N^{\mathcal{D}}.$$

It is easy to check (just apply the definition of the cotensor product) that:

$$\rho_N^{\mathcal{D}} \circ \varepsilon_N = (N \otimes_S \hat{\Phi}) \circ (i \otimes_R S).$$

Thus, we have:

$$\begin{aligned} (i \otimes_R S) \circ \xi_N \circ \varepsilon_N &= (N \otimes_S \hat{\Psi}) \circ \rho_N^{\mathcal{D}} \circ \varepsilon_N \\ &= (N \otimes_S \hat{\Psi}) \circ (N \otimes_S \hat{\Phi}) \circ (i \otimes_R S) = i \otimes_R S. \end{aligned}$$

Since $i \otimes_R S$ is a monomorphism, we conclude that $\xi_N \circ \varepsilon_N = FGN$. \square

Examples 3.23. Consider the adjunction $(F = \bullet \otimes_R S, G = \bullet \square_{\mathcal{D}}(S \otimes_R \mathcal{C}))$ between the categories $\mathcal{M}_R^{\mathcal{C}}$ and $\mathcal{M}_S^{\mathcal{D}}$. Observe first that the category $\mathcal{M}^{\mathcal{C}} = \mathcal{M}_R$ in the case where $\mathcal{C} = R$.

1) Let \mathcal{C} and \mathcal{D} be coalgebras over a field \mathbb{K} and $(\Phi, \varphi) = (\Phi, \mathbb{K})$. Then F is the corestriction of coscalars functor, and $G = \bullet \square_{\mathcal{D}} \mathcal{C}$. By Theorem 3.21, F is naturally full if and only if $\eta_{\mathcal{C}} : \mathcal{C} \rightarrow GFC = \mathcal{C} \square_{\mathcal{D}} \mathcal{C}$, $\eta_{\mathcal{C}}(c) = \sum c_{(1)} \square_{\mathcal{D}} c_{(2)}$ cosplits as a \mathcal{C} -bicomodule map. Since

$$(\Phi \square_{\mathcal{D}} \mathcal{C}) \circ \eta_{\mathcal{C}} = \mathcal{C} = (\mathcal{C} \square_{\mathcal{D}} \Phi) \circ \eta_{\mathcal{C}},$$

we have that $\Phi \square_{\mathcal{D}} \mathcal{C} = \mathcal{C} \square_{\mathcal{D}} \Phi$ and hence $\varepsilon_{\mathcal{C}}(c)c' = c\varepsilon_{\mathcal{C}}(c')$ for any $c, c' \in \mathcal{C}$ (recall that we have a canonical isomorphism $\mathcal{D} \square_{\mathcal{D}} \mathcal{C} \rightarrow \mathcal{C}$, $d \square_{\mathcal{D}} c \mapsto \varepsilon_{\mathcal{D}}(d)c$). In view of Proposition 3.13, this last property means that the forgetful functor $\mathcal{M}_{\mathbb{K}}^{\mathcal{C}} \rightarrow \mathcal{M}_{\mathbb{K}}$ is naturally full. Now, if $\mathcal{C} \neq 0$, there is a $c \in \mathcal{C}$ such that $\varepsilon_{\mathcal{C}}(c) \neq 0$. Since \mathbb{K} is a field, we can put $z := \varepsilon_{\mathcal{C}}(c)^{-1}c$ so that $\varepsilon_{\mathcal{C}}(z) = 1$. By Corollary 3.15, the functor $G = \bullet \otimes_{\mathbb{K}} \mathcal{C} : \mathcal{M}_{\mathbb{K}} \rightarrow \mathcal{M}_{\mathbb{K}}^{\mathcal{C}}$ is naturally full and, by Corollary 3.14, \mathcal{C} has a \mathbb{K} -ring structure by means of a ring homomorphism $\xi : \mathbb{K} \rightarrow \mathcal{C}$ such that $\xi \circ \varepsilon_{\mathcal{C}} = \mathcal{C}$. Since \mathbb{K} is a field, ξ is an isomorphism.

By Theorem 3.22, G is naturally full if and only if the \mathcal{D} -bicomodule map

$$\hat{\Phi} : \mathbb{K} \otimes_{\mathbb{K}} \mathcal{C} \otimes_{\mathbb{K}} \mathbb{K} \rightarrow \mathcal{D}, \quad \hat{\Phi}(k \otimes_{\mathbb{K}} c \otimes_{\mathbb{K}} k') = k\Phi(c)k'$$

splits as a \mathcal{D} -bicomodule map. Since $\mathbb{K} \otimes_{\mathbb{K}} \mathcal{C} \otimes_{\mathbb{K}} \mathbb{K} \simeq \mathcal{C}$, this is equivalent to Φ splitting as a \mathcal{D} -bicomodule map.

2) Let $(\Phi, \varphi) = (\varphi, \varphi)$, that is, $\mathcal{C} = R$ and $\mathcal{D} = S$. In this case the adjunction (F, G) reduces to extension and restriction of scalars:

$$F = \bullet \otimes_R S : \mathcal{M}_R \rightarrow \mathcal{M}_S ; G : \mathcal{M}_S \rightarrow \mathcal{M}_R.$$

By Theorem 3.21, F is naturally full if and only if $\eta_R : R \rightarrow GFR = R \otimes_R S$, $\eta_R(r) = r \otimes_R 1_S$, cosplits in ${}_R \mathcal{M}_R$. Since $l_S : R \otimes_R S \rightarrow S$, $l_S(r \otimes_R s) = \varphi(r)s$ is an isomorphism and $l_S \circ \eta_R = \varphi$, this is equivalent to φ cosplitting in ${}_R \mathcal{M}_R$, that is, there exists an $E \in {}_R \text{Hom}(S, R)_R$ such that $\varphi \circ E = S$. We have

therefore recovered condition 2-2) of Proposition 3.1.

By Theorem 3.22, G is naturally full if and only if the S -bimodule map $\hat{\varphi} : S \otimes_R R \otimes_R S \rightarrow S$, $\hat{\varphi}(s \otimes_R r \otimes_R s) = s\varphi(r)s'$, splits in ${}_S\mathcal{M}_S$. Since $\hat{\varphi} \circ (S \otimes_R l_s^{-1}) = \varepsilon_S$, the counit of the Sweedler coring $S \otimes_R S$, this is equivalent to ε_S splitting in ${}_S\mathcal{M}_S$. Since $\tau : S \rightarrow S \otimes_R S$, $\tau(s) = 1_S \otimes_R s$ is a retraction of ε_S (i.e. $\varepsilon_S \circ \tau = S$), we can conclude that G is naturally full if and only if ε_S is injective. So we have recovered condition (5) of Theorem 1.1.

3) Let $(\Phi, \varphi) = (\varepsilon_C, R)$, that is $\mathcal{D} = S = R$. The adjoint pair (F, G) is now the adjoint pair from Proposition 3.13.

By Theorem 3.21, F is naturally full if and only if $\eta_C = \Delta_C : \mathcal{C} \rightarrow GFC = \mathcal{C} \otimes_R \mathcal{C}$ cosplits in ${}_R\mathcal{M}_R^{\mathcal{C}}$. Since $(\varepsilon_C \otimes_R \mathcal{C}) \circ \Delta_C = \mathcal{C} = (\mathcal{C} \otimes_R \varepsilon_C) \circ \Delta_C$ this is equivalent to Δ_C being surjective, and we recover condition (3) of Proposition 3.13-2.

By Theorem 3.22, G is naturally full if and only if the R -bimodule map $\hat{\varepsilon}_C : R \otimes_R \mathcal{C} \otimes_R R \rightarrow R$, $\hat{\varepsilon}_C(r \otimes_R c \otimes_R r') = r\varepsilon_C(c)r'$, splits in ${}_R\mathcal{M}_R$. Since $R \otimes_R \mathcal{C} \otimes_R R \cong \mathcal{C}$, this is equivalent to ε_C splitting in ${}_R\mathcal{M}_R$: we recover condition (3) of Proposition 3.13-1.

Remark 3.24. Let $(R, \mathcal{C})_\psi$ be an entwining structure over a commutative ring K . It was shown in [1, Proposition 2.2] that $R \otimes_K \mathcal{C}$ has a coring structure in such a way that the category of comodules over $R \otimes_K \mathcal{C}$ is isomorphic to the category of entwined modules $\mathcal{M}_R^{\mathcal{C}}(\psi)$. Every morphism $(\varphi, \Theta) : (R, \mathcal{C})_\psi \rightarrow (S, \mathcal{D})_\phi$ of entwining structures induces a coring homomorphism $(\varphi \otimes_K \Theta, \varphi) : R \otimes_K \mathcal{C} \rightarrow S \otimes_K \mathcal{D}$. Applying Theorems 3.21 and 3.21, we can give necessary and sufficient conditions for the faithful fullness of the functors in the adjoint pair $(F = \bullet \otimes_R S, G = \bullet \square_{\mathcal{D}} \mathcal{C})$ between $\mathcal{M}_R^{\mathcal{C}}(\psi)$ and $\mathcal{M}_R^{\mathcal{D}}(\phi)$.

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